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# A study about Fourier series: Mathematical and graphical models and application in electric current and square oscillations

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***Abstract*— The present work aims to study the Fourier series with the intention of creating a didactic and understandable text. There are several ways to introduce the Fourier series, the simplest and most pedagogical way, at least for beginners in the subject, is to explore intuitively the basic idea through graphics. This analysis facilitates the understanding of the methodology behind the series and the Fourier. The mathematical formulation will be a simple extension of the physical (geometric) image presented by the graphics. The popular saying goes that a figure is worth a thousand words. Mathematicians, in turn, argue that an equation is worth a thousand figures. Maybe that is why in Fourier's analysis books, mathematical formulation comes first, followed by the graphics.*

## I. INTRODUCTION

The theory of Fourier series began marked by the attempt to solve heat conduction problems in a bar. In the mid-eighteenth century a big problem arose that helped both mathematical analysis and other areas, including physics. The problem was to write a certain function given as an infinite series of sine and cosine functions, more specifically a periodic function.

Many physical phenomena related to celestial mechanics, particle mechanics, wave, ballistics, heat transmission, and electrical circuits are translated into differential equations whose differential and integral calculus is not enough to solve them, arising the need for something more sophisticated, called Fourier equations.

The development and creation of this series is due to Jean Baptiste Joseph Fourier (1766-1830), French physicist and mathematician. Fourier used it to solve the problem of heat conduction, which he explains in his work *Théorie Analytique de la Chaleur* in 1822, as the theory of heat diffusion in solid bodies.

According to KOHAUPT and LEE (2015), the Fourier series was originated in telecommunication engineering, specifically in the connection with the digitalization of data (signals) in order to transmit them.

Different from the current days where signals are digitalized, formerly, signals were transmitted in the superposition of continuous periodic time functions, in the form of waves.

Although Fourier's proposition that any function (continue or discontinuous) could be written as an infinite sum of the trigonometric functions cosine and sine had been written down earlier, his contribution was ground-breaking by the degree of attention given to the convergence of the functions (FOLKERTS et. al, 2020).

Still, according to AMORIM et. al (2019), although the work of Fourier had been heavily criticised by Lagrange, Laplace and Legendre, the fact that Fourier have observed that discontinuous functions are the sum of infinite series it was a great advance, for it explicit the shape of the series that represents this functions.

The applications of Fourier series are the most varied. In the matter of electric current and oscillations, Fourier series are vital to approximate a periodic waveform in electronics and electrical circuits. For the modern technology and engineering, is by the use of this series that it is possible to decompose periodic signals into sum of infinite trigonometrically series in sine and cosine terms (ANUMAKA, 2012).

In this work, we approach the Fourier series in three sections. In the first section, we deal with the subjects related to the definition of series, even and odd functions, periodic functions, as well as their properties. In the second, Fourier's trigonometric coefficients are determined, giving some examples of functions written in the form of the Fourier series. Finally, in the third section, we talk about some applications using odd and even function, and cases where the functions are neither even nor odd.

## II. METHODOLOGY

In this chapter, we will study the definition of limits, derivatives, integrals, periodic functions, even functions, odd functions, sequences, series, their properties and their involvement with the series under study. We will now move on to formal settings.

### 2.1 Limits

We say that a function $f$ (x) has limit 'k' when $x \to a$, if for all $\varepsilon > 0$ there is $\delta > 0$ as demonstrated by MUNEM and FOLIS (1882):

$$0 < |x - a| < \delta \rightarrow |f(x) - k| < \varepsilon \qquad (1)$$

Such 'k' number when it exists is unique and will be indicated by:

$$\lim_{x \to a} f(x) = k \qquad (2)$$

#### 2.1.1 Limits properties

For $f$ (x) and g (x) functions of a real variable and K' belonging to IR. We consider:

$$\lim_{x \to a} f(x) = k \qquad (3)$$

and

$$\lim_{x \to a} g(x) = k \qquad (4)$$

Therefore:

i. $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + lim g(x) = k_1 + k_2;$

ii. $\lim_{x \to a}[f(x) . g(x)] = \lim_{x \to a} f(x). lim g(x) = k_1 * k_2;$

iii. $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{k_1}{k_2}, desde\ que\ k_2 \neq 0;$

iv. $\lim_{x \to a}[K^{`} * f(x)] = K^{`} . \lim_{x \to a} f(x) = k^{`} * k_1 .$

All demonstrations here omitted can be seen in GUIDORIZZI (2001).

### 2.2 Derivatives

Among limits, there is a special called derivatives, which is our next object of study. Be $f$ a function and k a point of your domain. The limit:

$$\lim_{x \to a} \frac{f(x+h) - f(x)}{h} \qquad (5)$$

When it exists and is finite, it is named derivative and is indicated by $f'$(k). Therefore:

$$f^{`}(k) = \lim_{x \to a} \frac{f(x+h) - f(x)}{h} \qquad (6)$$

If the limit above exists, we say that such function is differentiable at the given point.

#### 2.2.1 Derivatives Properties

Be f and g derivable in k and 'c' a constant. So we have to:

i. $(f + g)'(k) = f'(k) + g'(k)$

ii. $(f * g)'(k) = f'(k) * g(k) + f(k) * g'(k)$

iii. $[f(k) \div g(k)] = [f'(k)\, g(k) - f(k)\, g'(k)] \div [(g(k))^2]$, as long as $g(k) \neq 0$

iv. $(c\, f)'(k) = c * f'(k)$

All demonstrations here omitted can be seen in GUIDORIZZI (2001).

### 2.3 Integral

Given a function $f$ defined in a range [a,b] and k is a given number. The sum bellow is the Reimam sum:

$$\sum_{i=1}^{n} f(c_i)\Delta X_i, \quad onde\ c_i \in [X_i, X_{i-1}] \qquad (7)$$

When doing $(máx\Delta x_1) \rightarrow 0$, naturally the sum will tend to K, then we will have the following:

$$\lim_{máx\,\Delta X_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta X_i = K \qquad (8)$$

By setting the limit, given $\varepsilon > 0$ there is $\delta > 0$ that it only depends on $\varepsilon$:

$$|(\sum_{i=1}^{n} f(c_i)\Delta X_i) - K| < \varepsilon \qquad (9)$$

Such number K is indicated by:

$$\int_a^b f(x)dx \qquad (10)$$

So we refer to the equation below as the defined integral from *f* to [a,b].

$$K = \lim_{máx\,\Delta X_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta X_i = \int_a^b f(x)\,dx \qquad (11)$$

2.3.1 Integral properties

Be *f*, g integral in [a,b], and A a constant:

i. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

ii. $\int_a^b A * f(x)dx = A\int_a^b f(x)\,dx$

iii. $If\ f(x) \geq 0\ in\ [a,b], therefore\ \int_a^b f(x)dx \geq 0$

iv. If C ∈ [a,b] and *f* integrable in [a,c] and in [c,d], therefore:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

2.3.2 Fundamental Theorem of Calculus

Be f integral in [a,b], if F is a primitive of *f* in [a,b] then:

$$\int_a^b f(x)\,dx = F(a) - F(b) \qquad (12)$$

Such theorem is only used in the case where the integral is said to be defined.

2.3.3 Integração por partes

Be *f* and g defined and derivable in a range I. Note that:

$$(f * g)'(x) = f'(x) * g(x) + f(x) * g'(x) \qquad (13)$$

$$f(x) * g'(x) = (f * g)'(x) - f'(x) * g(x) \qquad (14)$$

$$\int_a^b f(x) * g`(x)dx = \int_a^b [(f * g)`(x) - f`(x) * g(x)]\,dx \qquad (15)$$

$$\int_a^b g`(x) * f(x)dx = f(x) * g(x) - \int_a^b f`(x) * g(x)dx \qquad (16)$$

Doing u=f(x) and v=g(x):

du = f' (x) dx and dv = g' (x) dx

Therefore, integration by parts is:

$$\int_a^b udu = uv - \int_a^b vdu \qquad (17)$$

2.4 Numeric Sequence

A sequence is a function whose domain is the set of natural numbers, that is, it is an endless succession of numbers.

For all n ∈ N, traditionally, a sequence is written as:

$$X_n = (X_0, X_1, X_2, \ldots, X_i, \ldots)$$

A sequence can converge or diverge. If the sequence converges to a real number ''k'' we say that it is represented by a limit, that is:

$$\lim_{x\to\infty} X_n = k$$

2.4.1 Propriedade de Sequências

The same operating rules used with function limits are valid for number sequences. See section 2.3.1 of this chapter.

2.5 Séries Numéricas

Given a sequence (Xn) n∈N, which may converge or not, we say that an infinite series is the sum:

$$X_o + X_1 + X_2 + \ldots + X_n \ldots$$

With the compressed writing:

$$S_n = \sum_{n=1}^{\infty} X_n \qquad (18)$$

If the series converges, we say that the limit of its sum exists and will be represented by a real S number, or rather:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{n=1}^{\infty} X_n = S \qquad (19)$$

2.6 Periodic Functions

We will begin the study of periodic functions, something fundamental to continue the understanding about the series under study. We say that a function f: ℝ → ℝ is periodic period t if f (x+t) = f (x) for all real x (Ávila, 1999).

The function cos (x) it is periodic of period 2π because:

Cos (x + 2π) = cos (x) ∗ cos (2π) – sen (x) ∗ sen (2π)

With cos (2π) = 1 and sen (2π) = 0, it follows:

Cos (x + 2π) = cos (x) ∗ 1 – sen (x) ∗ 0 = cos (x)

Be f and g functions of given period t, then f + g and f ∗ g will also be periodic.

Demonstration:

(f + g)(x +t) = f (x + t) + g(x + t), because any functions enjoy the definition of sum of functions. By applying the periodic function definition, we have:

(f + g)(x+t )= f (x + t) + g(x + t) = f (x) + g(x) = (f + g)(x)

O que nos mostra que a soma de duas funções periódicas também e periódica. A outra demonstração será feita de modo análogo veja:

(fg) = (x + t) = f (x + t)g(x + t) = f(x)g(x) = (fg)(x)

Which shows us that the product of two periodic functions is also periodic.

If function f is periodic period t, then kf(x) is periodic of the same period, where k is a natural other than zero.

Demonstration:

Be f [k(x + t)] = f [(x + t) + (x + t) + (x + t) + ... + (x + t)] = f (x + t) + f(x + t)+ f (x + t) + ... f (x + t) = f (x) + f (x) + f(x) +...+ f (x) = kf (x)

Which completes our demonstration.

If f is differentiable and periodic from period t, we will show that the derived function f is also periodic for the same period.

Demonstration:

Note that:

$$f'(x+t) = \lim_{h\to 0}\frac{f[(x+t)+h]-f(x+t)}{h} = \lim_{h\to 0}\frac{f(x+t)-f(x)}{h} = f'(x)$$

Be f: ℝ → ℝ a periodic function of period t, and integral in any range, we will show that:

$$\int_a^{a+t} f = \int_0^t f \qquad (20)$$

Where *a* it is any real number fixed.

Demonstration

By the fundamental theorem of calculus, we have the following equation:

$$\varphi(x) = \int_a^{a+t} f = F(a+t) - F(a) = F(a) - F(a) = 0$$

As

$$\varphi(x) = F(a+t) - F(a) = F(a) - F(a)$$

Therefore, $\varphi`(x) = 0$ we conclude that $\varphi(x) = c$ it is constant and $\varphi(a) = \varphi(0)$.

So:

$$\int_a^{a+t} f = \int_0^t f \qquad (21)$$

2.7 Even and Odd functions

A function f: ℝ → ℝ is even when f (x) = f (–x), for all real and odd x when f (x) = -f (–x), for all real x.

For example, the function f (x) = $x^2$ is even, because f (–x) = $(–x)^2$ = $x^2$ = f(x).

The function f (x) = $x^3$ is odd, because f (-x) = $(-x)^3$ = -x $^3$= -f(x).

2.7.1 Propositions

Now let's study some properties of even and odd functions. They are:

i. The sum of two even functions is an even function, the sum of two odd functions is odd;
ii. The product of two real even functions is an even function, and the product of two odd functions is an even;
iii. The product of a real function odd by an even real function is odd;
iv. Be f a real even function. The defined integral of f in [-l,l] is twice the defined integral of f in [0,l];
v. Be f an odd and integrable real function, then the defined integral of f in [-l,l] is null;
vi. Every real function f (x) can be decomposed into the sum:

f (x) = fp (x) + fi (x)

Where, fp (x) is an even function and fi (x) is an odd function, defined respectively by:

fp (x) = (f (x) + f (-x) ) / 2

fi (x) = (f (x) - f (-x) ) / 2

Demonstration:

Let's now demonstrate each item mentioned earlier.

i. Be φ:R→R and ω:R→R even functions.

(φ+ω)(-x) = φ(-x) + ω(-x) = φ(x) + ω(x) = (φ + ω)( x)

ii. Be φ:R→R and ω:R→R even functions.

(φ*ω)(x) = φ(x) + ω(x) = φ(-x)*ω(-x) = (φ*ω)(-x)

iii. Be φ:R→R an even function and ω:R→R an odd function.

(φ*ω)(x) = φ(x) + ω(x) = φ(-x)*ω(-x) = (φ*ω)(-x)

iv. As we know we can decomminate the integration limits of any integral defined in a finite range, as follows:

$$\int_{-l}^{l} f = \int_{-l}^{l} f(x)\,dx = \int_{-l}^{0} f(x)dx + \int_{0}^{l} f(x)\,dx$$

Doing x = -y, you have:

$$-\int_{-l}^{0} f(-y)\,dy + \int_{0}^{l} f(x)\,dx = \int_{-l}^{0} f(x)dx + \int_{0}^{l} f(x)\,dx$$

Because f(x) is even. Like this:

$$\int_{-l}^{0} f(y)\,dy + \int_{0}^{l} f(x)\,dx = \int_{0}^{l} f(x)dx + \int_{0}^{l} f(x)dx = 2\int_{0}^{l} f(x)\,dx$$

Therefore:

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)dx$$

v. The demonstration will be made in a manner analogous to item (iv):

$$\int_{-l}^{l} f = \int_{-l}^{l} f(x)\,dx = \int_{-l}^{0} f(x)\,dx + \int_{0}^{l} f(x)\,dx$$

Doing x = -y, you have:

$$-\int_{-l}^{0} f(-y)dy + \int_{0}^{l} f(x)dx = \int_{-l}^{0} f(y)dy + \int_{0}^{l} f(x)dx = -\int_{0}^{l} f(x)dx + \int_{0}^{l} f(x)dx$$

Therefore:

$$\int_{-l}^{l} f(x)\,dx = 0$$

vi. Note that if f(x) is pair, you have:

$$f_p(x) = \frac{2f_p(x)}{2} = \frac{f_p(x)+f_p(x)}{2} = \frac{f_p(x)+f_p(-x)}{2} = \frac{f(x)+f(-x)}{2}$$

Because, fp (x) = f (x) = f (-x), by definition. If f(x) is odd, it has – if:

$$f_i(x) = \frac{2f_i(x)}{2} = \frac{f_i(x)+f_i(x)}{2} = \frac{f_i(x)-f_i(-x)}{2} = \frac{f(x)-f(-x)}{2}$$

Because, fi (x) = f(x)= -f (-x), by definition. From this, we conclude that:

$$f_p(x)+f_i(x) = \frac{f_p(x)-f_i(x)}{2} + \frac{f_i(x)+f_i(-x)}{2} = \frac{f(x)-f(-x)}{2} + \frac{f(x)+f(-x)}{2}$$

$$\frac{f_p(x)+f(-x)+f(x)-f(-x)}{2} = \frac{f(x)+f(x)}{2} = \frac{2f(x)}{2} = f(x)$$

Therefore:

$$f(x) = f_p(x) + f_i(x)$$

## III. FOURIER SERIES

A Fourier series follows the form:

$$\frac{a_0}{2} + a_1\cos(x) + b_1 sen(x) + a_2\cos(2x) + b_2\, sen(2x) + \cdots$$

Or otherwise:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n sen(nx)] \tag{22}$$

It's called the Fourier series, with n=1,2,3,4,..., where:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \tag{23}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,sen(nx)dx \tag{24}$$

They are called Fourier coefficients, of the trigonometric series which will be explained in more detail. As we know a series can diverge or converge. If the series presented earlier converge sum, it will be a periodic function f(x) of period 2π, because sen (nx) and cos (nx), are periodic functions of period 2π. So that f (x+t) = f (x).

### 3.1 Determination of the trigonometric coefficients of Fourier

Be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n sen(nx)]$.

According to BUTKOV (1988) we have:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \tag{25}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \tag{26}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{sen}(nx)dx \tag{27}$$

#### 3.1.1 Proposition

To demonstrate the Fourier coefficients we will make use of the following proposition. Suppose the functions are integrable in a range I where the series is evenly fit. So:

$$\int_I \left[\sum_{n=1}^{\infty} f_n(x)\right] dx = \sum_{n=1}^{\infty}\left[\int_I f_n(x)\right]dx \tag{28}$$

Demonstration:

Note that:

$$\int_l \left[\sum_{n=1}^{\infty} f_n(x)\right] dx = \int_l [f_1(x) + f_2(x) + \cdots + f_n(x)]dx$$

$$\int_l \left[\sum_{n=1}^{\infty} f_n(x)\right] dx = \int_l f_1(x)\,dx + \int_l f_2(x)\,dx + \cdots + \int_l f_n(x)dx$$

$$\int_l \left[\sum_{n=1}^{\infty} f_n(x)\right] dx = \sum_{n=1}^{\infty} [\int_l f_n(x)]dx$$

This demonstrates the theorem mentioned above. Now, let's deduce Fourier's coefficients. When a function is written in the form of a Fourier series, we have the following:

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n sen(nx)] \quad (29)$$

The Fourier series can be integrated term-to-term in the range (-π.π) due to preposition 3.1.1. So we have:

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_o}{2} dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n sen(nx)]\,dx \quad (30)$$

Now, let's calculate each integral separately.

$$\int_{-\pi}^{\pi} \frac{a_o}{2} dx = \frac{a_0}{2} \int_{-\pi}^{\pi} dx = \frac{a_o}{2} = a_o\,\pi$$

$$\int_{-\pi}^{\pi} a_n \cos(nx)dx = \frac{a_n}{n} \int_{-\pi}^{\pi} \cos(u)du = \frac{-a_n}{n} [sen(n\pi) - sen(-n\pi)]$$

$$\int_{-\pi}^{\pi} a_n \cos(nx)\,dx = \frac{-a_n}{n} [sen(n\pi) - sen(-n\pi)] = 0$$

$$\int_{-\pi}^{\pi} b_n \mathrm{sen}(nx)dx = \frac{b_n}{n} \int_{-\pi}^{\pi} \mathrm{sen}(u)du = \frac{b_n}{n} [cos(n\pi) - cos(-n\pi)]$$

$$\int_{-\pi}^{\pi} a_n \cos(nx)\,dx = \frac{b_n}{n} [cos(n\pi) - cos(n\pi)] = 0$$

Going back to:

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_o}{2} dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n sen(nx)]dx$$

We have:

$$\int_{-\pi}^{\pi} f(x)\,dx = a_o\pi \Leftrightarrow a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx \quad (31)$$

To determine Fourier's coefficients $a_n$ and $b_n$, we will use the following auxiliary integrals, i.e., orthogonally relationships. If n and k are whole and if n≠k, you have:

$$\int_{-\pi}^{\pi} \cos(nx)\cos(nk)dx = 0$$

$$\int_{-\pi}^{\pi} \cos(nx)\,\mathrm{sen}(nk)dx = 0$$

$$\int_{-\pi}^{\pi} \mathrm{sen}(nx)\,\mathrm{sen}(nk)dx = 0$$

<u>Demonstration:</u>

We will show the results presented by the auxiliary integrals.

$$\int_{-\pi}^{\pi} \cos(nx)\cos(nk)dx = \int_{-\pi}^{\pi} \frac{1}{2} [\cos(n+k) + \cos(n-k)]dx$$

$$\frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(n+k)\,\mathrm{dx} + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-k)\right] dx$$

$$u = (n+k)x \Rightarrow \frac{du}{dx} = (n+k) \Rightarrow du = (n+k)\,dx$$

and

$$w = (n-k)x \Rightarrow \frac{dw}{dx} = (n-k) \Rightarrow dw = (n-k)dx$$

Therefore:

$$\frac{1}{2(n+k)} \int_{-\pi}^{\pi} \cos(u)\,du + \frac{1}{2(n-k)} \int_{-\pi}^{\pi} \cos(w)\,dw$$

Integrating, we have:

$$\frac{1}{2(n+k)} sen(u)]_{-\pi}^{\pi} - \frac{1}{2(n-k)} sen(w)]_{-\pi}^{\pi} =$$

$$\frac{1}{2(n+k)} sen(n+k)]_{-\pi}^{\pi} - \frac{1}{2(n-k)} sen(n-k)]_{-\pi}^{\pi} =$$

$$\{\frac{1}{2(n+k)} [sen(n+k)\pi] - \frac{1}{2(n+k)} [sen(n+k)(-\pi)] -$$

$$\frac{1}{2(n-k)}[sen(n-k)\pi] - \frac{1}{2(n-k)}[sen(n-k)(-\pi)]\} =$$

$$\{\frac{1}{2(n+k)}[sen(n+k)\pi] - \frac{1}{2(n+k)}[sen(n+k)(\pi)] -$$

$$\frac{1}{2(n-k)}[sen(n-k)\pi] - \frac{1}{2(n-k)}[sen(n-k)(\pi)]\}$$

$$\{0+0\} = 0$$

$$\int_{-\pi}^{\pi} \cos(nx) sen(nk) dx = \int_{-\pi}^{\pi} \frac{1}{2}[\cos[(n+k)x] - sen[(n-k)x]]dx =$$

$$\frac{1}{2}\int_{-\pi}^{\pi} \cos[(n+k)x]\,dx - \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{sen}[(n-k)x]\,dx =$$

$$u = (n+k)x \Rightarrow \frac{du}{dx} = (n+k) \Rightarrow du(n+k)dx$$

$$w = (n-k)x \Rightarrow \frac{dw}{dx} = (n-k) \Rightarrow dw(n-k)dx$$

$$\frac{1}{2(n+k)}\int_{-\pi}^{\pi} \mathrm{sen}(u)\,du + \frac{1}{2(n-k)}\int_{-\pi}^{\pi} \mathrm{sen}(w)\,dw$$

$$\frac{1}{2(n+k)}[-cos(u)]_{-\pi}^{\pi} - \frac{1}{2(n-k)}[-cos(w)]_{-\pi}^{\pi} =$$

$$\frac{1}{2(n+k)}[-cos(n+k)x]_{-\pi}^{\pi} - \frac{1}{2(n-k)}[-cos(n-k)]_{-\pi}^{\pi} =$$

$$\{\frac{1}{2(n+k)}[cos(n+k)\pi] - \frac{1}{2(n+k)}cos[(n+k)(-\pi)] -$$

$$\frac{1}{2(n-k)}[cos(n-k)\pi] - \frac{1}{2(n-k)}cos[(n-k)(-\pi)]\} =$$

$$\{\frac{1}{2(n+k)}[cos(n+k)\pi] - \frac{1}{2(n+k)}[cos(n+k)(\pi)] -$$

$$\frac{1}{2(n-k)}[cos(n-k)\pi] - \frac{1}{2(n-k)}[cos(n-k)(\pi)]\}$$

$$\{0+0\} = 0$$

$$\int_{-\pi}^{\pi} sen(nx) sen(nk) dx = \int_{-\pi}^{\pi} sen[(n+k)x]dx + \int_{-\pi}^{\pi} sen[(n-k)x]dx$$

$$u = (n+k)x \Rightarrow \frac{du}{dx} = (n+k) \Rightarrow du = (n+k)dx$$

$$w = (n-k)x \Rightarrow \frac{dw}{dx} = (n-k) \Rightarrow dw = (n-k)dx$$

$$\int_{-\pi}^{\pi} sen(u)du + \int_{-\pi}^{\pi} sen(w)dw =$$

$$\frac{1}{2(n+k)}\int_{-\pi}^{\pi} \mathrm{sen}(u)\,du + \frac{1}{2(n-k)}\int_{-\pi}^{\pi} \mathrm{sen}(w)\,dw =$$

$$\frac{1}{2(n+k)}[-\cos(u)]_{-\pi}^{\pi} + \frac{1}{2(n-k)}[-\cos(w)]_{-\pi}^{\pi} =$$

$$\frac{1}{2(n+k)}[-\cos(n+k)x)]_{-\pi}^{\pi} + \frac{1}{2(n-k)}[-\cos(n-k)x]_{-\pi}^{\pi} =$$

$$\{\frac{1}{2(n+k)}[\cos(n+k)\pi] - \frac{1}{2(n+k)}\cos[(n+k)(-\pi)] -$$

$$\frac{1}{2(n-k)}[\cos(n-k)\pi] + \frac{1}{2(n-k)}\cos[(n-k)(-\pi)]\} =$$

$$\{\frac{1}{2(n+k)}[\cos(n+k)\pi] - \frac{1}{2(n+k)}\cos[(n+k)(\pi)] -$$

$$\frac{1}{2(n-k)}[\cos(n-k)\pi] - \frac{1}{2(n-k)}\cos[(n-k)(\pi)]\}$$

$$\{0+0\} = 0$$

Now if n≠k, we have:

$$\int_{-\pi}^{\pi} cos^2(nx)dx$$

$$\int_{-\pi}^{\pi} cos(nx)\,sen(nx)dx$$

$$\int_{-\pi}^{\pi} sen^2(nx)\,dx \qquad (32)$$

Deducing equation (32) we have:

$$\int_{-\pi}^{\pi} cos^2(nx)dx = \int_{-\pi}^{\pi} \frac{1}{2}[1 + \cos(2nx)]dx$$

$$\int_{-\pi}^{\pi} \frac{1}{2}dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(2nx)\,dx$$

$$\frac{1}{2}x + \frac{1}{4n}sen(2nx)]_{-\pi}^{\pi} = [\frac{1}{2}(\pi - (-\pi) + \left[\frac{1}{4n}sen(2n\pi) + \frac{1}{4n}sen(2n(\pi))\right]$$

$$[\frac{1}{2}(\pi+\pi) + \left[\frac{1}{4n}sen(2n\pi) + \frac{1}{4n}sen(2n(\pi))\right]$$

$$[\frac{1}{2}(2\pi) + \left[\frac{1}{4n}sen(2n\pi) + \frac{1}{4n}sen(2n\pi)\right]$$

$$\pi + 2\frac{1}{4n}sen(2n\pi)$$

$$\pi + \frac{1}{2n}sen(2n\pi)$$

With 0 = sem (2π) = sen (2nπ) and π + (1/2n) ∗ 0 = π, therefore:

$$\int_{-\pi}^{\pi} cos^2(nx)\,dx = \pi \qquad \int_{-\pi}^{\pi} sen^2(nx)\,dx = \pi \tag{33}$$

Deducing the equation (33) we have:

$$\int_{-\pi}^{\pi} cos(nx)\,sen(nx)\,dx =$$

$$u = sen(nx) \Rightarrow \frac{du}{dx} = -ncos(nx) \Rightarrow du = -ncos(nx)dx$$

$$-\int_{-\pi}^{\pi} cos(nx)u\frac{du}{ncos(nx)} = \frac{1}{4n}[-sen(2n\pi) - sen(2n(\pi)\int_{-\pi}^{\pi} sen^2(nx)dx - \frac{1}{n}\int_{-\pi}^{\pi} udu =$$

$$-\frac{u^2}{2n}]_{-\pi}^{\pi} = -\frac{sen^2(nx)}{2n}]_{-\pi}^{\pi} = \left\{-\frac{sen^2(nx)}{2n} - \left[-\frac{sen^2(n(-\pi))}{2n}\right]\right\}$$

$$\left\{-\frac{sen^2(nx)}{2n} + \frac{sen^2(n\pi)}{2n}\right\} = 0$$

Therefore:

$$\int_{-\pi}^{\pi} cos(nx)\,sen(nx)\,dx = 0 \tag{34}$$

Deducing the equation (34) we have:

$$\int_{-\pi}^{\pi} sen^2(nx)\,dx = \int_{-\pi}^{\pi} \frac{1}{2}[1 - \cos(2nx)]dx$$

$$\int_{-\pi}^{\pi} \frac{1}{2}dx + \int_{-\pi}^{\pi} \frac{\cos(2nx)}{2}dx$$

$$\frac{1}{2}\{(\pi - (-\pi)] - +\frac{1}{4n}[-sen(2n\pi) + sen(2n(-\pi))]$$

$$\frac{1}{2}2\pi - +\frac{1}{4n}[-sen(2n\pi) - sen(2n(\pi))$$

$$\pi + \frac{1}{4n}[-2sen(2n\pi)$$

$$\pi - \frac{1}{2n}2sen(2n(\pi)$$

As 0=sen(2π) = sem (2nπ) e π - (1/2n) ∗ 0= π, therefore:

Now let's determine the coefficients $a_n$ and $b_n$. Suppose f(x) is periodic, can be represented in the form of Fourier.

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n sen(nx)]$$

Multiplying both members by cos (kx), with k 0, we have:

$$f(x)\cos(kx) = \frac{a_o}{2}\cos(kx) + \sum_{n=1}^{\infty}[a_n \cos(nx)\cos(kx) + b_n sen(nx)\cos(nk)]$$

By Proposition (3.1.1), we have that the Fourier trigonometric series can be integrated term to term.

$$\int_{-\pi}^{\pi} f(x)cos(kx)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos(kx)\,dx +$$

$$\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}[a_n \cos(nx)\,\cos(kx) + b_n sen(nx)\cos(nk)]dx$$

With n = k:

$$\int_{-\pi}^{\pi} f(x)cos(kx)dx = \int_{-\pi}^{\pi}\frac{a_0}{2}\cos(kx)dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi}[a_n \cos^2(nx) + b_n sen(nx)\cos(nk)]\,dx$$

$$\int_{-\pi}^{\pi} f(x)cos(kx)\,dx = \int_{-\pi}^{\pi} \cos^2(nx)\,dx = a_k\pi$$

Therefore:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,cos(kx)\,dx \tag{35}$$

Now, multiplying the two members of the equality by without (nx) and integrating again over the range (-π, π), one obtains:

$$f(x)sen(kx)\,dx = \frac{a_0}{2}\,\text{sen}(kx)\,dx +$$

$$\sum_{n=1}^{\infty}[a_n \cos(nx)sen(kx) + b_n sen(nx)\mathrm{sen}(nk)]dx$$

$$\int_{-\pi}^{\pi} f(x)sen(kx)\,dx = b_k \int_{-\pi}^{\pi} \mathrm{sen}^2(kx)\,dx = b_k\pi$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{f(x)\,sen}(kx)\,dx$$

3.2 The Fourier theorem

If an f(x) function is periodic or not written in the form of Fourier then it converges evenly to the midpoint of the function at each point. That is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n sen(nx)] \tag{36}$$

Where:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \tag{37}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \tag{38}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{sen}(nx)dx \tag{39}$$

3.2.1 Examples of functions developed in the form of Fourier

Any function can be expressed in the form of Fourier. If the given function is periodic we have nothing to say but if it is not we will have to define it in a certain subset of your domain. Here are some examples of Fourier series development.

Example 1: A periodic 2π period f(x) function is set as follows:

$$f(x) = x; -\pi \leq x \leq \pi$$

In this way, we have:

$$a_0 = \int_{-\pi}^{\pi} xdx = \frac{1}{\pi}\frac{x^2}{2}]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right] = \frac{1}{\pi}0 = 0 \tag{40}$$

Therefore, $a_0$=0. Calculating ak, we obtain:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} xcos(kx)dx$$

By doing the part-by-part integration technique:

$$uv - \int vdu$$

$$u = x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$$

$$dv = \cos(kx)\,dx = x \Rightarrow \int dv = \int \cos(kx)\,dx$$

$$v = -\frac{1}{k}sen(kx)$$

$$-\frac{xsen(kx)}{k}]_{-\pi}^{\pi} - \int \frac{1}{k}sen(kx)\,dx =$$

$$-\frac{xsen(kx)}{k}]_{-\pi}^{\pi} - \int \frac{1}{k^2}cos(kx)]_{-\pi}^{\pi} = 0$$

Calculating $b_k$:

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\mathrm{sen}(kx)\,dx = \frac{1}{k}\{\frac{xcos(kx)}{k}]_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi} \cos(kx)\,dx =$$

$$(-1)^{k+1}\frac{2}{k}$$

$$f(x) = 2\left[\frac{sen(x)}{1} - \frac{sen(2x)}{2} + \frac{sen(3x)}{3} - \frac{sen(4x)}{4} + \cdots (-1)^{k+1}\frac{sen(kx)}{k} \ldots\right]$$

$$f(x) = \sum_{n=1}^{\infty}(-1)^{k+1}\frac{sen(kx)}{k} \tag{41}$$

Example 2: Be f(x) = cos(3t). Calculate the coefficients $a_0$, $a_n$, $b_n$.

Calculating $a_0$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(3t)\,dt \Rightarrow a_0 = \frac{1}{3\pi}\int_{-\pi}^{\pi} cos(w)dw \tag{42}$$

$$a_0 = \frac{1}{3\pi}\int_{-\pi}^{\pi} cos(w)\,dw]_{-\pi}^{\pi} \Rightarrow a_0 = \frac{1}{3\pi}[-sen(3\pi) - sen(-3\pi)]$$

$$a_0 = \frac{1}{3\pi}[-sen(3\pi) + sen(3\pi)] = 0$$

Because the sin(x) function is odd i.e.; -sin(x) = sin(-x).

Calculating $a_n$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(3t)\cos(nt)dt \tag{43}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}[\cos(3-n)\,t+\cos(3+n)]tdt$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(3-n)\,tdt + \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(3+n)\,tdt$$

$$a_n = \frac{-1}{2\pi(3-n)} sen(3-n)t]_{-\pi}^{\pi} - \frac{1}{2\pi(3+n)} sen(3+n)t]_{-\pi}^{\pi}$$

Applying the integration limits, you have:

$$a_n = \frac{-1}{2\pi(3-n)}[sen[(3-n)\pi] - sen[(3-n)(-\pi)]] -$$

$$\frac{-1}{2\pi(3+n)}[sen[(3+n)\pi] - sen[(3+n)(-\pi)]]$$

$$a_n = \frac{-1}{2\pi(3-n)}[sen[(3-n)\pi] + sen[(3-n)\pi]] -$$

$$\frac{-1}{2\pi(3+n)}[sen[(3+n)\pi] + sen[(3+n)\pi]]$$

$$a_n = \frac{1}{\pi(3-n)} sen[(3-n)\pi] - \frac{1}{\pi(3+n)} sen[(3+n)\pi]$$

Therefore:

$$a_n = \frac{2nsen(n\pi)}{\pi(3-n)(3+n)}$$

Calculating $b_n$:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(3t)\, sen(nt)dt \tag{44}$$

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\text{sen}(3-n)t + sen(3+n)t]dt$$

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} sen(3-n)\,tdt + \frac{1}{2\pi}\int_{-\pi}^{\pi} sen(3+n)\,tdt$$

$$b_n = \frac{1}{2\pi(3-n)}\cos(3-n)\,t]_{-\pi}^{\pi} + \frac{1}{2\pi(3+n)}\cos(3+n)\,t]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{2\pi(3+n)}\cos[(3+n)\pi] - \frac{1}{2\pi(3+n)}\cos(3+n)(-\pi) -$$

$$\frac{1}{2\pi(3-n)}\cos[(3-n)\pi] - \frac{1}{2\pi(3-n)}\cos(3-n)(-\pi)$$

Because the function cos(x) is even, that is, cos (x) = cos (-x). We conclude that $b_n = 0$.

Therefore the Fourier series representing the given function will be:

$$f(t) = \sum_{n=1}^{\infty}\frac{2nsen(\pi n)}{\pi(3-n)(3+n)}\cos(nt) \tag{45}$$

## IV. FOURIER SERIES APPLICATION

In order to present the applications of the Fourier series, we first have to analyse graphically the functions even and odd and the ones that are neither even nor odd.

4.1 Graphical Examples

To start, we will analyze three examples: the first includes an even function, the second an odd function, and the third a function neither even nor odd.

Example 1: Be the par and periodic function of period $2\pi$:

$$\begin{cases} f(t) = t^2, -\pi < t \le \pi \\ f(t+2\pi) = f(t) \end{cases}$$

We demonstrate graphically that this function can be synthesized in a sum of cosines of the type:

$$f(t) \sim \frac{\pi^2}{3} - 4\left(cost - \frac{1}{4}cos2t + \frac{1}{9}cos3t - \frac{1}{16}cos4t + \frac{1}{25}cos5t - \cdots\right)$$

Each cosine term in the series is called harmonic. The first harmonic is called fundamental harmonic. Therefore, cos(t) is the fundamental harmonic; cos(2t), the second harmonic; cos(3t), the third harmonic and so on. It should be noted that the amplitude of each harmonic is equal to $(-1)n * 4/n2$. So the amplitude of the twentieth harmonic will be 1/100. Also note that the angular frequency, defined by the number of period per unit of time and expressed in radian per second, of each harmonic grows linearly with the order of the harmonics.

Figure 1 illustrates the approximation of a par function by the sum of cosines.

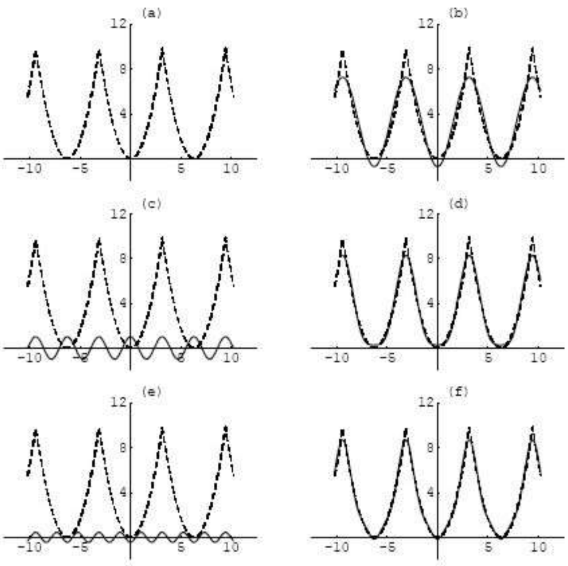


*Fig. 1 approximation of a par function by the sum of cosines*

In Figure 1, graph (a) represents the function f(t); graph (b) shows the first two terms of the sum overlapped with the graph of function f (t):

$$\frac{\pi^2}{3} - 4\cos(t) \tag{46}$$

In (c), the graphs of the function of the second harmonic, cos (2t); in (d) the graph of the first three terms overlapped with the graph of the f(t) function:

$$\frac{\pi^2}{3} - 4\cos(t) + \cos(2t) \tag{47}$$

The graph (e) plots the function and the third harmonic, and finally in (f) the graph of the first four terms overlaps with that of the function.

Examining these graphs, it is observed that with only three harmonics one already has a reasonable approximation of function. Note that the amplitude of each additional harmonic decreases while the frequency gradually grows. Thus, the small details in the function graphics are filled in by the higher-order harmonics.

Intuitively, we can say that as the number of harmonics grows, the approximation of the function, by the sum, becomes more and more perfect.

As it is well remembered by SILVA and PINEHIRO (2012), the analysis of harmonic signals using the time domain in most cases is of complex resolution and with many integrals. An easier way to analyse signals is by using the frequency domain using the Fourier Transform. The Fourier Transform (equation 48) is a useful tool, because through it differential and integral equations are reduced to simple algebraic equations. Its use is of great importance to know analysis of the total energy of the time series.

$$f(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt \tag{48}$$

Applying the Fourier Transform, we have a graph usually called the Energy Spectrum or Frequency with on the axis of the abscesses having the frequency and axis of the ordered energy. Each bar of a graphic is a phasor with certain frequency and amplitude. Using the frequency spectrum graph, we can represent a non-periodic signal as the sum of its phase components (SILVA and PINHEIRO, 2012).

Example 2: The function we just analysed is a periodic even function. Now, we're going to investigate an odd periodic function. Therefore, the following periodic function of period $2\pi$:

$$\begin{cases} f(t) = -1, -\pi < t \leq \pi \\ f(t) = 1, \quad 0 < t \leq \pi \\ f(t + 2\pi) = f(t) \end{cases}$$

Similarly to the previous example, we draw the graphics of the function and the first harmonics. As this is an odd function let's now use sine harmonics. Such an approximation and provided by:

$$f(t) \sim \frac{4}{\pi}\left(sint + \frac{1}{3}sin3t + \frac{1}{5}sin5t + \frac{1}{7}sin7t + \cdots\right)$$

Note that the amplitudes of the harmonics are expressed by $4/(2n + 1)\pi$, with $n \geq 1$ .

The graphic analysis in Figure 2 follows exactly the same script used in Figure 1. It is observed that with only three harmonics it is not possible to adjust this function as well as the previous even function. This means that convergence is slower in this case. In other words, much more harmonics will be needed to obtain the same degree of approximation obtained in the previous case. The reason for this is that this function is more uncompromising than the previous one because it is discontinued, whereas the previous one is a continuous function.

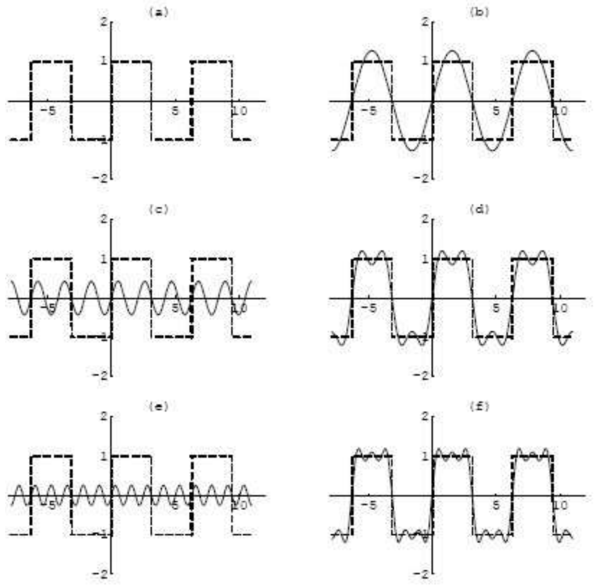

*Fig. 2: Approximation of an odd function by sum of sines.*

It is still surprising that even though it is a discontinuous function, it can be approximated by continuous functions (sine) with the degree of precision one wants. And only increase the number of harmonics.

At first mathematicians were very reluctant to accept this fact proposed by Fourier. At that time even the concept of function was not well defined yet. In fact, much of the advance of mathematical analysis in the 19th and early 20th centuries is due to this problem of convergence in the Fourier series.

Example 3: In the last two examples we have graphically seen that a pair function can be approximated by cosines and an odd function by sines. What if the function is neither even nor odd? We will soon see that any function (periodic or not) can be decomposed into an even component and an odd component. Therefore, a periodic function neither even nor odd can be approximated by cosines (even component adjustment) and sines (odd component adjustment) simultaneously. Be therefore the periodic function of period $2\pi$:

$$\begin{cases} f(t) = e^t & -\pi < t \leq \pi, \\ f(t+2\pi) = f(t) \end{cases}$$

Similarly to the two previous examples, we show graphically that this function is approximated by a sum of sines and cosines of the type:

$$f(t) \sim 2\sinh 1 \left[ \begin{array}{l} \frac{1}{2} - \frac{1}{1+\pi^2}(\cos t - \pi \sin t) + \frac{1}{1+4\pi^2}(\cos 2t - 2\pi \sin 2t) \\ -\frac{1}{1+9\pi^2}(\cos 3t - 3\pi \sin 3t) + \frac{1}{1+16\pi^2}(\cos 4t - 4\pi \sin 4t) - \ldots \end{array} \right]$$

With harmonic amplitude ($\cos(nt) - n\pi.\sin(nt)$) equal to $2\sinh(1)/(1+n^2.\pi^2)$.

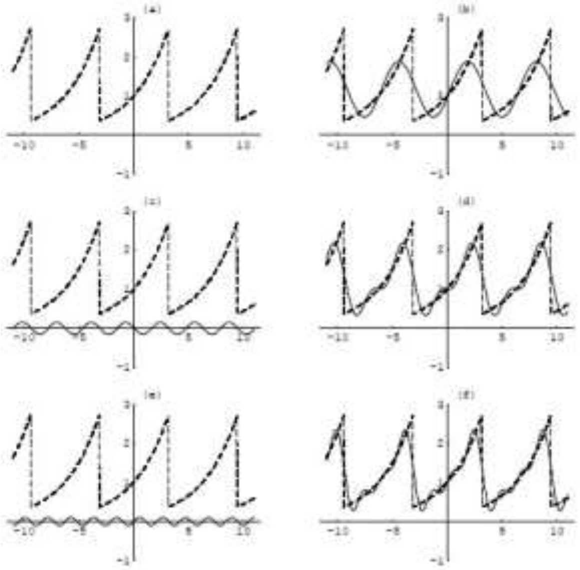

*Fig. 3 Approximation of a function (neither even nor odd) by sum o cosines and sines.*

As in the two previous examples, it is observed in Figure 3 that as the number of harmonics increases, the adjustment between the function graph and the sum of harmonics gradually improves.

A very important detail that should be highlighted in the three examples we have just seen is that in the first and third examples a constant term appears in front of the first harmonic: in the first example, the value $\pi 2/3$ is obtained, and in the third example sinh(1). In the second example, apparently this constant term does not exist. The point of this is that if the function is anti-symmetrical with respect to the axis of the abscissas, the constant term is identically zero. This is the case with the function of the second example. The functions of the first and third examples do not have this type of symmetry, and therefore the constant term is different from zero.

The constant term shifts the function vertically up or down. It is called the DC component of the signal, what stands for direct current. When you add continue current to an alternating current, you get a vertical offset of the alternating current. Hence the reason for the abbreviation DC. We'll get back to that later.

In the last example, the fact that the function was unfolded in an even component and another odd component was fundamental to be able to simultaneously use the sine and cosine in the approximation of the function. This is a general rule, that is, any function can effectively be decomposed into two components, one even and one odd.

4.2 Application in Electric Current

An alternating current i(t) = A.sen(x) has passed through a complete wave rectifier, which transmits the absolute (instantaneous) value of the current. We will show that the Fourier series that express is given by HALLIDAY (2003):

$$\frac{2A}{\pi}-\frac{4A}{\pi}\sum_{N=2,4,6,\ldots}^{\infty}\frac{\cos(nx)}{n^2-1} \tag{49}$$

If it were a half-wave rectifier it would be:

$$\frac{A}{2}+\frac{A}{2}sen(nx)-\frac{2A}{2}\sum_{N=1,3,5,\ldots}^{\infty}\frac{\cos(nx+x)}{n^2-n} \tag{50}$$

However, first we must define the function that will be written under function of Fourier:

$$\begin{cases} f(t)=A.sen(x), & 0\le t\le\pi,\\ f(t+2\pi)=f(t), & \pi<t\le 2\pi \end{cases}$$

Calculating the coefficient $a_0$:

$$a_0=\frac{1}{\pi}\int_0^{\pi}Asen(x)\,dx-\frac{1}{\pi}\int_{\pi}^{2\pi}Asen(x)\,dx$$

$$a_0=\frac{A}{\pi}(-\cos(x)]_0^{\pi}+\frac{A}{\pi}\cos(x)]_{\pi}^{2\pi})$$

$$a_0=\frac{4A}{\pi}$$

Calculating the coefficient $b_n$:

$$b_n=\frac{1}{\pi}\int_0^{\pi}Asen^2(nx)\,dx-\frac{1}{\pi}\int_{\pi}^{2\pi}Asen(x)\,sen(nx)\,dx$$

For n=1, you have:

$$b_1=\frac{1}{\pi}\int_0^{\pi}Asen^2(x)dx-\frac{1}{\pi}\int_{\pi}^{2\pi}Asen^2(x)dx$$

$$b_1=\frac{A}{2\pi}\int_0^{\pi}(1-\cos(x))dx-\frac{A}{2\pi}\int_{\pi}^{2\pi}(1+\cos(2x))dx$$

$$b_1=\frac{A}{2\pi}]_0^{\pi}+\frac{sen(2x)}{2}]_0^{\pi}-x]_{\pi}^{2\pi}-\frac{sen(2x)}{2}]_{\pi}^{2\pi}$$

$$b_1=\frac{A}{2\pi}(\pi-2\pi+\pi)=0$$

If we continue, we will see that all terms in sine are zero. Calculating the coefficient $b_n$:

$$b_n=\frac{1}{\pi}\int_0^{\pi}Asen(x)\cos(nx)\,dx-\frac{1}{\pi}\int_{\pi}^{2\pi}Asen(x)\cos(nx)dx$$

Note that it is enough to solve only one of the above integrals, because the only difference between them are the limits of integration. We will call the first integral (I) and resolve through part-by-party integration.

$$I=\frac{1}{\pi}\int_0^{\pi}Asen(x)\cos(nx)\,dx=$$

$$\frac{1}{n}sen(x)sen(nx)-\int_0^{\pi}Asen(nx)\cos(x)\,dx=$$

$$\frac{1}{\pi}sen(x)sen(nx)-\frac{\cos(x)\cos(nx)}{n^2}+\frac{I}{n^2}$$

$$I\left(1-\frac{1}{n^2}\right)=\frac{1}{n}sen(x)sen(nx)-\frac{\cos(x)\cos(nx)}{n^2}$$

Applying the integration limits we will have:

$$I=\frac{n^2}{n^2-1}[\frac{1}{n}sen(x)sen(nx)-\frac{1}{n^2}\cos(x)\cos(nx)]_0^{\pi}$$

$$I=\frac{n^2}{n^2-1}\left[-\frac{1}{n^2}(-1)^n-\frac{1}{n^2}\right]$$

$$I=\frac{n^2}{n^2-1}[(-1)^{n+1}-1]$$

Where, n=1,2,3,4, ... . So the result of the integral is:

$$I=\frac{1}{\pi}\int_0^{\pi}Asen(x)\cos(nx)\,dx=\frac{-2}{\pi(n^2-1)}$$

Applying the integration limits of the second integral and summing both we have:

$$a_n=\frac{-4A}{\pi(n^2-1)}$$

Where, n=2, 4, 6, 8, ... . Therefore the function representing the function will be:

$$g(x)=\frac{2A}{\pi}-\frac{4A}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{\cos(nx)}{n^2-1} \tag{51}$$

The function written in the form of Fourier will be:

$$f(x) = \begin{cases} Asen(x), se\ 0 \le x \le \pi \\ 0, sen \le x \le 2\pi \end{cases}$$

Calculating the coefficient $a_0$:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} Asen(x)\,dx + \frac{1}{\pi}\int_0^{\pi} 0dx$$

$$a_0 = \frac{A}{\pi}\int_0^{\pi} sen(x)\,dx =$$

$$\frac{-Acos(x)}{\pi}]_0^{\pi} = \frac{2A}{\pi}$$

Calculating the coefficient $b_n$:

$$b_n = \frac{1}{\pi}\int_0^{\pi} Asen(x)sen(nx)\,dx$$

Where n=1, you have:

$$b_1 = \frac{1}{\pi}\int_0^{\pi} sen^2(x)dx = \frac{1}{2}\pi\int_0^{\pi}(1+\cos(x))dx = \frac{1}{2} + \frac{sen(2\pi)}{4} = \frac{1}{2}$$

Therefore, the first term in sine will be:

$$\frac{A}{2}sen(x)$$

If we continue we will see that all terms in sine will cancel out, except the first.

Calculation of $a_n$:

$$a_n = \frac{1}{\pi}\int_0^{\pi} Asen(x)\cos(nx)\,dx$$

By the previous demonstration we have the result of this integral.

$$a_n = \frac{-2}{\pi(n^2-1)}$$

Doing n=k+1 in $a_n$:

$$a_k = \frac{-2}{\pi(k^2-2k)}$$

Therefore:

$$\frac{A}{2} + \frac{A}{2}sen(nx) - \frac{2A}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{\cos(nx+x)}{n^2+n}$$

ARAÚJO and MÁRQUEZ (2015) examines a linear differential equation that represents a simples electric circuit of the type resistor inductor (RL), of which the electrical tension is expressed by a function E (t) impossible to be resolved analytically. In this particular case, the solution have not been expressed by a combination of elementary functions due to the nature of the function. However, by the use of the Fourier Transform and complex integration, it was possible to obtain an explicit expression involving the model parameters presented earlier in this paper, to represent the intensity of the electrical current of the examined circuit as a function of time.

### 4.3 Application in square oscillations

Another Fourier application to analyse is the square oscillation that can occur in an electronic circuit designated to guide the rise of the pulses. Suppose the oscillation is defined by:

$$f(x) = \begin{cases} 0, se -\pi \le x \le \pi \\ h, se, 0 \le x \le \pi \end{cases}$$

Using the equations for the calculation of Fourier coefficients, we have:

$$a_n = \frac{1}{\pi}\int_0^{\pi} hdt = h$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} hcos(nt)dt = 0$$

$$b_n = \frac{h}{\pi}\frac{1}{n}$$

If n is odd:

$$b_n = 0$$

If n is even, the result is:

$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left[\frac{sen(x)}{1} + \frac{sen(3x)}{3} + \frac{sen(5x)}{5} + \cdots\right]$$

Note that all terms in cosine disappear in the range (-π.π), that the terms that will represent the interpolated function will all be sine, and that this type of application is related to the following reasoning: As our electric current is of the sine type, but there are machines that do not work with this current (a good example of this are the pulse motors), the current arrives at the machine with a sine behaviour and the rectifier transforms it into the current that the machine needs. This transformation was described above through the Fourier series.

A Fourier series can also be represented as a spectrum of frequency. In a graphic, amplitude vs. frequency, as proposed in Figure 4 by ANDRADE (2003), the relative amplitude of every frequency of square wave are represented by simple lines.

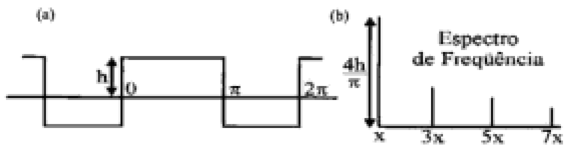


*Fig. 4 (a) square wave representation followed by its frequency spectrum (b) (ANDRADE, 2003).*

It is also demonstrated by Andrade (2003), how simple it is to explore theoretical aspects of physics and mathematics utilizing oscillating circuits in laboratory. In his experiment, it was obtained the Fourier coefficients applying to the circuit a tension with the shape of a square wave and frequency $\omega_0$. Although a square wave is applied in the circuit, the electric current circulating is sinusoidal with maximum intensity in this shape. Therefore, it is conclude that a square wave is composed by an infinite series of harmonic waves with determinate frequencies and in consequence is possible to create a Fourier spectrum for the wave.

For the numeric solution of different equations in the context of analysis of oscillators amplitude, whenever is applied the Fourier transform the results a simple and precise (CARLOS EDUARDO, 2018). In the case of resonance the Fourier transform is again an important tool.

## V. CONCLUSION

We hope that the reader has understood the process of expanding a defined function in a range in an even or odd periodic function. The choice of the series in terms of sines (odd expansion) or cosines (even expansion) depends on the nature of the problem that one wants to solve.

It should be in mind that the periodic expansion of a function restricted to a range serves only as a support to use the Fourier series technique; and that once the expansion is done, the values outside the definition range of the function become irrelevant, and therefore disposable. Only the approximation of the function, by the Fourier series, within the definition range of the function is of interest, the rest, I repeat, is disposable. It is good to know that, in practice, non-periodic functions defined in a range are more important and frequent than the periodic functions themselves.

So far, the impression one has is that any periodic function allows to be approximated by a series of Fourier. Is this true? The definitive answer to this question is subtle and is outside the objectives of this work. However, we can ensure that all periodic functions found here, in particular in electromagnetism can be developed by Fourier series. Technically, they meet Direchlet's conditions, which are sufficient conditions for a periodic function to be expanded in Fourier series.

Periodicity is one of the prerequisites for Fourier serial development. This leads us to ask the following question: If the function is non-periodic, is it still possible to speak in Fourier series? Roughly speaking, the answer is affirmative. There are two possibilities to be analysed: (a) functions restricted to a finite range of the line and (b) not-periodic functions defined in any straight or semi-straight.

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