

BASIC STEFFENSEN'S METHOD OF HIGHER-ORDER CONVERGENCE

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Abstract: In this paper, we introduce a new analog of a variant of Steffensen's method of fourth-order convergence for solving non-linear equations based on the q-deference operator.

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1. INTRODUCTION

Finding the zeros of a nonlinear equation, $f(x) = 0$, is a classical problem of numerical analysis. Analytic methods for solving such equations rarely exist, and therefore, one can hope to obtain only approximate solutions by relying on iteration methods. For a survey of the most important algorithms, some excellent textbooks are available, see [4, 8, 10]. The classical Newtons method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots \quad (1.1)$$

Being quadratically convergent, Newton's method is probably the best known and most widely used algorithm. Time to time the method has been derived and modified in a variety of ways. One such method derived from Newton's method by approximating the derivative with non-derivative term of difference quotient is Steffensen's method [9, 11]. The method requires two evaluations of function and is quadratically convergent. The interesting iterative scheme is Steffensen's method that has the following form:

$$x_{n+1} = x_n - \frac{f^2(x_n)}{(f(x_n + f(x_n)) - f(x_n))}, \quad n = 0, 1, 2, \dots \quad (1.2)$$

In order to control the approximation of the derivative and the stability of the iteration, a Steffensen's type method has been proposed in [2], this approach is based on a better approximation to the derivative $f'(x_n)$ in each iteration. It has the following form:

$$x_{n+1} = x_n - \frac{f(x_n)}{(f(x_n + \alpha_n |f(x_n)|f(x_n)) - f(x_n))/\alpha_n |f(x_n)|f(x_n)}. \quad (1.3)$$

After that, the paper [1] has extended the above result on Banach spaces, obtained its local and semi-local convergence theorems, and made its applications on boundary-value problems by multiple shooting methods.

A family of fourth order methods free from any derivative, satisfying the highest convergence order were established in [12–14].

2. q -CALCULUS

In the following, q is a positive number, $0 < q < 1$. For $n \in \mathbb{N} = \{0, 1, \dots\}$, $k \in \mathbb{Z}^+ = \{1, 2, \dots\}$ and $a, a_1, \dots, a_k \in \mathbb{C}$, the q -shifted factorial, the multiple q -shifted factorial and the q -binomial coefficients are defined by

$$(a; q)_0 := 1, \quad (a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j), \quad (a_1, a_2, \dots, a_k; q)_n := \prod_{j=1}^k (a_j; q)_n, \quad (2.1)$$

and

$$\begin{bmatrix} a \\ 0 \end{bmatrix}_q := 1, \quad \text{and} \quad \begin{bmatrix} a \\ n \end{bmatrix}_q := \frac{(1 - q^a)(1 - q^{a-1}) \cdots (1 - q^{a-n+1})}{(q; q)_n}, \quad (2.2)$$

respectively. The limit, $\lim_{n \rightarrow \infty} (a; q)_n$, is denoted by $(a; q)_\infty$. Moreover $(a; q)_n$ has the representation, cf. [5],

$$(a; q)_n = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} a^k. \quad (2.3)$$

The q -Gamma function, [5, 6], is defined by

$$\Gamma_q(z) := \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z}, \quad z \in \mathbb{C}, \quad |q| < 1, \quad (2.4)$$

where we take the principal values of q^z and $(1 - q)^{1-z}$. In particular

$$\Gamma_q(n + 1) = \frac{(q; q)_n}{(1 - q)^n}, \quad n \in \mathbb{N}.$$

Let $\mu \in \mathbb{C}$ be fixed. A set $A \subseteq \mathbb{C}$ is called a μ -geometric set if for $x \in A$, $\mu x \in A$. Let f be a function defined on a q -geometric set $A \subseteq \mathbb{C}$. The q -difference operator is defined by the formula

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx}, \quad x \in A - \{0\}. \quad (2.5)$$

If $0 \in A$, we say that f has q -derivative at zero if the limit

$$\lim_{n \rightarrow \infty} \frac{f(xq^n) - f(0)}{xq^n}, \quad x \in A \quad (2.6)$$

exists and does not depend on x . We then denote this limit by $D_q f(0)$. The q -integration of F. H. Jackson [7] is defined for a function f defined on a q -geometric set A to be

$$\int_a^b f(t) d_q t := \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \quad a, b \in A, \quad (2.7)$$

where

$$\int_0^x f(t) d_q t := \sum_{n=0}^{\infty} xq^n (1 - q) f(xq^n), \quad x \in A, \quad (2.8)$$

provided that the series converges. A function f which is defined on a q -geometric set A , $0 \in A$, is said to be q -regular at zero if

$$\lim_{n \rightarrow \infty} f(xq^n) = f(0), \quad \text{for every } x \in A.$$

The rule of q -integration by parts is

$$\int_0^a g(x)D_q f(x) d_q x = (fg)(a) - \lim_{n \rightarrow \infty} (fg)(aq^n) - \int_0^a D_q g(x)f(qx) d_q x. \quad (2.9)$$

If f, g are q -regular at zero, the $\lim_{n \rightarrow \infty} (fg)(aq^n)$ on the right hand side of (2.9) will be replaced by $(fg)(0)$. The two variable polynomial $\varphi_n(x, a)$, $x, a \in \mathbb{C}$, are defined to be

$$\varphi_0(x, a) := 1, \quad \varphi_n(x, a) := \begin{cases} x^n (a/x; q)_n, & x \neq 0, \\ (-1)^n q^{\frac{n(n-1)}{2}} a^n, & x = 0. \end{cases} \quad (2.10)$$

In [3], Annaby and Mansour gave q -Taylor series in the following forms

$$f(x) = \sum_{k=0}^{n-1} \frac{D_q^k f(a)}{\Gamma_q(k+1)} \varphi_k(x, a) + \frac{1}{\Gamma_q(n)} \int_a^x \varphi_{n-1}(x, qt) D_q^n f(t) d_q t. \quad (2.11)$$

$$f(x) = \sum_{k=0}^{n-1} (-1)^k q^{-\frac{k(k-1)}{2}} \frac{D_q^k f(aq^{-k})}{\Gamma_q(k+1)} \varphi_k(a, x) + \frac{1}{\Gamma_q(n)} \int_{aq^{-n+1}}^x \varphi_{n-1}(x, qt) D_q^n f(t) d_q t, \quad (2.12)$$

3. A q -STEFFENSEN-SECANT METHOD

In the following we set $e_n = x_n - a$, $e_n^* = y_n - a$, $z_n = x_n + qf(x_n)$, $y_n = x_n - f(x_n)/f[x_n, z_n]$, where $f[a, b] = \frac{f(a)-f(b)}{a-b}$,

$$A = \frac{D_q f(a)}{\Gamma_q(2)} + \frac{a(1-q)D_q^2 f(a)}{\Gamma_q(3)} + \frac{a^2(1-q)^2(1+q)D_q^3 f(a)}{\Gamma_q(4)}, \quad (3.1)$$

$$B = \frac{D_q^2 f(a)}{\Gamma_q(3)} + \frac{a(1-q)(2+q)D_q^3 f(a)}{\Gamma_q(4)}, \quad (3.2)$$

and

$$C = \frac{D_q^3 f(a)}{\Gamma_q(4)}. \quad (3.3)$$

Now, we state and prove our q -Steffensen-secant Theorem with fourth order convergence.

Theorem 3.1. *Let $f : \mathcal{D} \rightarrow \mathbb{R}$ be a real-valued function with a root $a \in \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}$, and let x_0 be closed enough to a . If $D_q^k(x)$, $k = 1, 2, 3$ exist, and $D_q(a) \neq 0$, then*

$$x_{n+1} = y_n - \frac{f[x_n, y_n] - f[z_n, y_n] + f[z_n, x_n]}{f^2[x_n, y_n]} f(y_n), \quad n \in \mathbb{N}, \quad (3.4)$$

is fourth-order convergent, and satisfies the following error equation

$$e_{n+1} = A^{-1}B(1+qA) \left[A^{-1}C(1+qA) - A^{-2}B(3+2qA+2q^2A^2) \right] e_n^4 + O(e_n^5), \quad n \in \mathbb{N}. \quad (3.5)$$

Proof. Using the Taylor expansion in (2.11), we have

$$\begin{aligned}
 f(x_n) = & \\
 & \frac{D_q f(a)}{\Gamma_q(2)}(x_n - a) + \frac{D_q^2 f(a)}{\Gamma_q(3)}(x_n - a)(x_n - qa) + \\
 & \frac{D_q^3 f(a)}{\Gamma_q(4)}(x_n - a)(x_n - qa)(x_n - q^2 a) + \frac{1}{\Gamma_q(4)} \int_a^{x_n} \varphi_3(a, qt) D_q^4 f(t) d_{qt}.
 \end{aligned} \tag{3.6}$$

Rearranging the above equation again gives:

$$f(x_n) = Ae_n + Be_n^2 + Ce_n^3 + O(e_n^4), \tag{3.7}$$

that is

$$\begin{aligned}
 f(z_n) = f(x_n + qf(x_n)) = & \\
 & \frac{1}{\Gamma_q(4)} \int_a^{x_n + qf(x_n)} \varphi_3(a, qt) D_q^4 f(t) d_{qt} + \frac{D_q f(a)}{\Gamma_q(2)}(x_n - a + qf(x_n)) \\
 & + \frac{D_q^2 f(a)}{\Gamma_q(3)}(x_n - a + qf(x_n))(x_n - qa + qf(x_n)) + \\
 & \frac{D_q^3 f(a)}{\Gamma_q(4)}(x_n - a + qf(x_n))(x_n - qa + qf(x_n))(x_n - q^2 a + qf(x_n)) \\
 & = O(e_n^4) + \frac{D_q f(a)}{\Gamma_q(2)}(e_n + qf(x_n)) \\
 & + \frac{D_q^2 f(a)}{\Gamma_q(3)}(e_n + qf(x_n))(e_n + qf(x_n) + a(1 - q)) + \\
 & \frac{D_q^3 f(a)}{\Gamma_q(4)}(e_n + qf(x_n))(e_n + qf(x_n) + a(1 - q))(e_n + qf(x_n) + a(1 - q^2)) \\
 & = A(e_n + qf(x_n)) + B(e_n + qf(x_n))^2 + C(e_n + qf(x_n))^3 + O(e_n^4).
 \end{aligned} \tag{3.8}$$

Thus,

$$\begin{aligned}
 f(z_n) = & \\
 & A[1 + qA]e_n + B[1 + 3qA + q^2 A^2]e_n^2 + \\
 & \left[C[1 + 4qA + 3q^2 A^2 + q^3 A^3] + 2qB^2[1 + qA] \right] e_n^3 + O(e_n^4).
 \end{aligned} \tag{3.9}$$

Moreover,

$$\begin{aligned}
 f[z_n, x_n] = \frac{f(x_n + qf(x_n)) - f(x_n)}{qf(x_n)} & \\
 = A + B[2 + qA]e_n + \left[C[3 + 3qA + q^2 A^2] + qB^2 \right] e_n^2 + O(e_n^3). &
 \end{aligned} \tag{3.10}$$

Therefore,

$$g(x_n) := \frac{f(x_n)}{f[z_n, x_n]} = O(e_n^4) + e_n - A^{-1}B[1 + qA]e_n^2 + \left[A^{-2}B^2[1 + qA][2 + qA] - qA^{-1}B^2 - A^{-1}C[2 + 3qA + q^2A^2] \right] e_n^3. \quad (3.11)$$

Consequently,

$$\begin{aligned} f(y_n) &= f(x_n - g(x_n)) = \\ &= \frac{D_q f(a)}{\Gamma_q(2)}(x_n - a - g(x_n)) + \frac{D_q^2 f(a)}{\Gamma_q(3)}(x_n - a - g(x_n))(x_n - qa - g(x_n)) \\ &+ \frac{D_q^3 f(a)}{\Gamma_q(4)}(x_n - a - g(x_n))(x_n - qa - g(x_n))(x_n - q^2a - g(x_n)) \\ &+ \frac{1}{\Gamma_q(4)} \int_a^{x_n - g(x_n)} \varphi_3(a, qt) D_q^4 f(t) d_q t \\ &= O(e_n^4) + \frac{D_q f(a)}{\Gamma_q(2)}(e_n - g(x_n)) + \\ &+ \frac{D_q^2 f(a)}{\Gamma_q(3)}(e_n - g(x_n))(e_n + qf(x_n) + a(1 - q)) + \\ &+ \frac{D_q^3 f(a)}{\Gamma_q(4)}(e_n - g(x_n))(e_n - g(x_n) + a(1 - q))(e_n - g(x_n) + a(1 - q^2)) \\ &= A(e_n - g(x_n)) + B(e_n - g(x_n))^2 + C(e_n - g(x_n))^3 + O(e_n^4). \end{aligned} \quad (3.12)$$

This means

$$f(y_n) = O(e_n^4) + B[1 + qA]e_n^2 - \left[A^{-1}B^2[1 + qA][2 + qA] - qB^2 - C[2 + 3qA + q^2A^2] \right] e_n^3, \quad (3.13)$$

and

$$e_n^* = O(e_n^4) + A^{-1}B[1 + qA]e_n^2 - \left[A^{-2}B^2[1 + qA][2 + qA] - qA^{-1}B^2 - A^{-1}C[2 + 3qA + q^2A^2] \right] e_n^3. \quad (3.14)$$

On the other hand

$$\begin{aligned} f[x_n, y_n] &= \frac{f(x_n) - f(y_n)}{g(x_n)} \\ &= A + Be_n + \left[C + A^{-1}B^2[1 + qA] \right] e_n^2 + O(e_n^3). \end{aligned} \quad (3.15)$$

Hence

$$f^2[x_n, y_n] = O(e_n^4) + A^2 + 2ABe_n + \left[2AC + B^2[3 + 2qA] \right] e_n^2 + \left[2BC + 2A^{-1}B^3[1 + qA] \right] e_n^3. \quad (3.16)$$

But

$$f[z_n, y_n] = \frac{f(z_n) - f(y_n)}{qf(x_n) + g(x_n)} = A + B(1 + qA)e_n + \left[C(1 + qA)^2 + A^{-1}B^2(1 + 4qA + 2q^2A^2) \right] e_n^2 + O(e_n^3). \quad (3.17)$$

So that

$$H(x_n) = \frac{f[y_n, x_n] - f[z_n, y_n] + f[z_n, x_n]}{f^2[y_n, x_n]} = A^{-1} + \left[A^{-2}C(1 + qA) - A^{-3}B(3 + 2qA + 2q^2A^2) \right] e_n^2 + \left[-2A^{-3}BC(2 + qA) + A^{-4}B^2(5 + 3qA + 4q^2A^2) \right] e_n^3 + O(e_n^4). \quad (3.18)$$

If we multiply $H(x_n)$ by $f(y_n)$ we get

$$H(x_n)f(y_n) = H(x_n)f[y_n, a]e_n^* = \left[1 + \left[A^{-1}C(1 + qA) - A^{-2}B(3 + 2qA + 2q^2A^2) \right] e_n^2 + \left[-2A^{-2}BC(2 + qA) + A^{-3}B^2(5 + 3qA + 4q^2A^2) \right] e_n^3 + O(e_n^4) \right] e_n^*. \quad (3.19)$$

Taking in consideration that x_{n+1} is nothing but $y_n - H(x_n)f(y_n)$ we get

$$x_{n+1} = y_n - H(x_n)f(y_n) = x_n - \left[1 + \left[A^{-1}C(1 + qA) - A^{-2}B(3 + 2qA + 2q^2A^2) \right] e_n^2 + \left[-2A^{-2}BC(2 + qA) + A^{-3}B^2(5 + 3qA + 4q^2A^2) \right] e_n^3 + O(e_n^4) \right] e_n^*. \quad (3.20)$$

Thus

$$e_{n+1} = \left[A^{-1}C(1 + qA) - A^{-2}B(3 + 2qA + 2q^2A^2) + O(e_n) \right] e_n^2 e_n^* = A^{-1}B[1 + qA] \left[A^{-1}C(1 + qA) - A^{-2}B(3 + 2qA + 2q^2A^2) \right] e_n^4 + O(e_n^5). \quad (3.21)$$

This completes the proof. □

In order to compare our new method with Steffensen's method, we give the following example.

Example: In this example we take

$$f(x) = \cos(x) - x.$$

The root of $f(x)$ is $a = 0.7390851332$. Then the sequence $\{x_n\}_n$

$$\begin{aligned}
 & x_{n+1} = y_n \\
 & - \frac{qf^2(x_n) \left[\left(\cos(y_n) - x_n \right) E_q(x_n) + qf^2(x_n) \right]}{\left[E_q(x_n) + f(x_n) \right] \left[\left(\cos(y_n) - \cos(x_n) \right) E_q(x_n) + qf^2(x_n) \right]} \\
 & + \frac{qf^4(x_n) \left[\left(\cos(y_n) - x_n \right) E_q(x_n) + qf^2(x_n) \right]}{\left[E_q(x_n) + f(x_n) \right] \left[\left(\cos(y_n) - \cos(x_n) \right) E_q(x_n) + qf^2(x_n) \right]^2},
 \end{aligned}$$

is fourth-order convergent, where

$$\begin{aligned}
 E_q(x_n) &= \cos \left(q \cos(x_n) + (1 - q)x_n \right) - (1 + q) \cos(x_n) + qx_n, \\
 y_n &= x_n - \frac{q \left(\cos(x_n) - x_n \right)^2}{E_q(x_n)}.
 \end{aligned}$$

Taking $x_0 = 0$, for $q = 0.5$, we find

	x_1	x_2	x_3	x_4
Our's	0.8617217519	0.7399567610	0.7390851885	0.7390851332
Steffensen's	2.175342650	0.76343368	0.7613122807	0.7595358304

Taking $x_0 = 1.1$, for $q = 0.9$, we find

	x_1	x_2	x_3	x_4
Our's	0.7063822168	0.7388491909	0.7390851206	0.7390851333
Steffensen's	0.8296038833	0.8040964255	0.7902498570	0.7814206993

Taking $x_0 = 1.35$, for $q = 0.001$, we find

	x_1	x_2	x_3	x_4
Our's	0.8144712303	0.74139713209	0.7390873914	0.7390851090
Steffensen's	0.7429374052	0.7428816874	0.7428275625	0.7427749629

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