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Time Series Water Wave Model Based on Energy Conservation

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Abstract*— *The time series water wave model comprises two equations: the water surface elevation equation and the water particle velocity equation. The elevation equation, derived from the continuity equation, captures changes in potential energy but lacks an energy source. The velocity equation describes changes in kinetic energy but does not include potential energy contributions. In this research, the energy deficiency in the elevation equation is addressed by combining it with the conservation of kinetic energy, while the velocity equation incorporates a driving force derived from the continuity equation. Both equations use a weighted Taylor series expansion to produce realistic wavelengths consistent with natural waves. The model is executed using a periodic solitary wave input, defined as a profile in which the trough and crest lie above the still water level. This input satisfies the initial conditions at the entry point, where the water surface elevation and its time derivative are zero. Shoaling and breaking analysis with the developed model yielded breaking parameters that agree with previous research.

I. INTRODUCTION

Time series models are widely applied in water wave analysis. The Boussinesq equation (1871) has been the most extensively developed and utilized. Subsequent refinements have been made by numerous scholars, including Peregrine (1967), Hamm and Madsen (1993), Nwogu (1993), Dingemans (1997), Johnson (1997), Madsen and Schaffer (1998), and Kirby (2003).

The Airy long-wave equation, originally introduced for tidal wave modeling (Dean, 1991), shares the same fundamental structure as the Boussinesq formulation and can therefore be regarded as a special case within the Boussinesq framework.

A well-known limitation of the Boussinesq equation is its applicability to only small wave amplitudes. To address this, Hutahaean (2005a) combined the continuity and energy conservation equations to produce a model capable of representing larger wave amplitudes for a given wave period. Later, Hutahaean (2025a) refined the convective acceleration term in the velocity equation, though without modifying the surface elevation equation, thereby advancing the time series model for use with larger wave amplitudes.

Despite these advances, time series models still face challenges in representing wave transformation from deep to shallow water. While shoaling effects can be reasonably simulated, wave breaking remains difficult to capture accurately.

Hutahaean (2024a, 2024b) developed a time series model based on the water surface elevation equation, formulated as a superposition of the continuity and kinetic energy conservation equations, without further refinement of the velocity equation. Both equations were derived using a weighted Taylor series, producing short wavelengths that closely approximate those observed in nature. However, the model remained limited in its ability to simulate wave breaking.

Another challenge in time series modeling concerns wave input. With sinusoidal input, the initial condition at the input point (zero) is not satisfied. According to the

Kinematic Free Surface Boundary Condition, the water particle velocity at this point should be zero, yet it is instead at its maximum. This inconsistency introduces errors, particularly in short-wave modeling, where vertical water particle velocity is a key variable.

The present research addresses these limitations by incorporating energy conservation into both the water surface elevation and velocity equations. The convective acceleration term in the velocity equation was further refined by interpreting it as a hydrodynamic force (Hutahaean, 2025a). To achieve wavelengths consistent with those found in nature, the formulation again employs a weighted Taylor series. The model proposed in this research employs a periodic solitary wave profile to ensure that both the water surface elevation and its differential are zero at the initial calculation point.

II. DEPTH AVERAGE VELOCITY, INTEGRATION COEFFICIENT AND TRANSFORMATION COEFFICIENT

In formulating the water surface elevation and horizontal water particle velocity equations, integration with respect to water depth is performed using the concept of depth-averaged velocity. Accordingly, both equations are expressed in terms of depth-averaged velocity. Since the formulation also involves surface particle velocity, this quantity must be transformed into its depth-averaged equivalent. Depth-averaged velocity is defined as the velocity at an elevation $z_0$below the still water level (Fig 1).

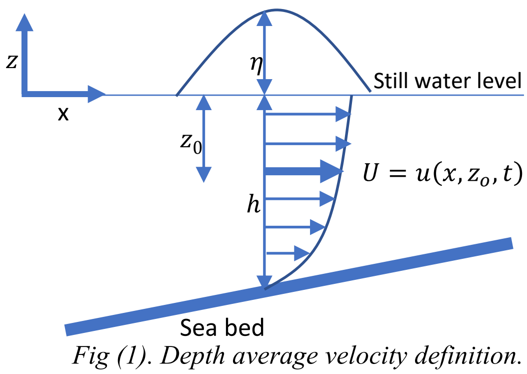

Fig (1). Depth average velocity definition.

From the velocity potential equation,

$$\phi(x, z, t) = 2\, G \cos kx \cosh k(h + z) \sin \sigma t \quad ...(1)$$

G is the wave constant, $k = \frac{2\pi}{L}$ is the wave number and L is the wavelength, $\sigma = \frac{2\pi}{T}$ is the angular frequency, T is the wave period.

Horizontal water particle velocity is,

$$u(x, z, t) = -\frac{\partial \phi}{\partial x}$$
$$= 2\, Gk \sin kx \cosh k(+z) \sin \sigma t \quad ...(2)$$

According to the definition, the depth-averaged horizontal water particle velocity is,

$$U = u(x, z_0, t) = 2\, Gk \sin kx \cosh k(h + z_0) \sin \sigma t \quad ....(3)$$

The vertical water particle velocity is expressed as,

$$w(x, z, t) = -\frac{\partial \phi}{\partial z}$$
$$= -2\, Gk \cos kx \sinh k(h + z) \sin \sigma t \quad ..(4)$$

Accordingly, the depth-averaged vertical water particle velocity is,

$$W = w(x, z_0, t)$$
$$= -2\, Gk \cos kx \sinh k(+z) \sin \sigma t \quad ....(5)$$

2.1. Integration coefficient and transformation coefficient.

a. Integration coefficient of water particle velocity β_u , β_{uu}, β_{uuu} and β_{ww}.

a.1. Integration coefficient β_u

The integration of the horizontal water particle velocity, as defined by Dean (1991) is

$$\int_{-h}^{\eta} u\, dz = \beta_u U D \quad ...(6)$$

$U = u(x, z_0, t)$ is the depth average velocity, total water depth $D = h + \eta$, h is the water depth towards still water level Fig (1), and β_u is the integration- coefficient. Thus, the integration coefficient equation of β_u is,

$$\beta_u = \frac{\int_{-h}^{\eta} u\, dz}{UD}$$

From (2) and (3),

$$\beta_u = \frac{\int_{-h}^{\eta} 2\, Gk \sin kx \cosh k(h + z) \sin \sigma t\ dz}{D\ 2\, Gk \sin kx \cosh k(h + z_0) \sin \sigma t}$$

Common terms in the numerator and denominator cancel, and upon evaluating the integral, one obtains,

$$\beta_u = \frac{\sinh k(h + \eta)}{kD\ \cosh k(h + z_0)} \quad ...(7)$$

From the wave number conservation relation (Hutahaean, 2023),

$$k(h + \eta) = kD = \theta\pi \quad ...(8)$$

Where θ is the deep water coefficient with $\tanh \theta\pi \approx 1$, therefore (7) is

$$\beta_u = \frac{\sinh \theta\pi}{\theta\pi\ \cosh kh(1 + \frac{z_0}{h})}$$

For $\varepsilon = \left|\frac{z_0}{h}\right|$

$$\beta_u = \frac{\sinh \theta\pi}{\theta\pi \, \cosh kh(1-\varepsilon)}$$

With the small amplitude approach, $kh = \theta\pi$

$$\beta_u = \frac{\sinh \theta\pi}{\theta\pi \, \cosh \theta\pi(1-\varepsilon)} \quad \dots(9)$$

a.2. Integration coefficient β_{uu}

Is defined as,

$$\beta_{uu} = \frac{\int_{-h}^{\eta} uu \, dz}{UUD} \quad \dots(10)$$

Substituting (2) and (3), and following the same procedure as in the derivation of β_u yields,

$$\beta_{uu} = \frac{\left(\frac{1}{2}\sinh 2\theta\pi + \theta\pi\right)}{2\theta\pi \, cosh^2\theta\pi(1-\varepsilon)} \quad \dots(11)$$

a.3. Integration coefficient β_{uuu}

Is defined as,

$$\beta_{uuu} = \frac{\int_{-h}^{\eta} uuu \, dz}{UUUD} \quad \dots(12)$$

Substituting (2) and (3), and following the same steps as in the derivation of β_u results,

$$\beta_{uuu} = \frac{\frac{1}{3}\sinh 3\theta\pi + 3\sinh\theta\pi}{8\theta\pi cosh^3\theta\pi(1-\varepsilon)} \quad ..(13)$$

a.4. Integration coefficient β_{ww}

Is defined as,

$$\beta_{ww} = \frac{\int_{-h}^{\eta} ww \, dz}{WWD} \quad \dots(14)$$

By substituting (4) and (5), and following the same procedure as in the derivation of β_u as follows

$$\beta_{ww} = \frac{\left(\frac{1}{2}\sinh 2\theta\pi - \theta\pi\right)}{2\theta\pi \, cosh^2\theta\pi(1-\varepsilon)} \quad \dots(15)$$

b. Transformation coefficient $\alpha_{u\eta}$ and $\alpha_{w\eta}$

The relationship between the horizontal surface water particle velocity and the horizontal depth-averaged water particle velocity is expressed as,

$$u_\eta = \alpha_{u\eta} U$$

u_η is the surface horizontal water particle velocity, $\alpha_{u\eta}$ is the transformation-coefficient,

$$\alpha_{u\eta} = \frac{\cosh\theta\pi}{\cosh\theta\pi(1-\varepsilon)} \quad \dots(16)$$

Similarly, the vertical transformation coefficient is obtained as,

$$\alpha_{w\eta} = \frac{\sinh\theta\pi}{\sinh\theta\pi(1-\varepsilon)} \quad \dots(17)$$

In this research, a value of $\theta = 1.00$, where $\tanh\theta\pi = 0.996272$ is close to 1.0. The corresponding values of the integration coefficients for θ is presented in Table (1).

Table (1). The Values of Integration Coefficient.

ε	β_u	β_{uu}	β_{uuu}	β_{ww}
0.37	0.99688	1.60363	1.66625	1.65154
0.371	0.99990	1.61335	1.68143	1.66236
0.372	1.00293	1.62314	1.69675	1.67325
0.373	1.00596	1.63298	1.71220	1.68421
0.374	1.00901	1.64287	1.72779	1.69525
0.375	1.01206	1.65283	1.74352	1.70637
0.376	1.01512	1.66284	1.75939	1.71756
0.377	1.01819	1.67291	1.77539	1.72882
0.378	1.02127	1.68304	1.79154	1.74017
0.379	1.02435	1.69323	1.80783	1.75159

The values of the integration coefficients employed in this research correspond to those for which β_u is closest to, or equal to, 1.0. As shown in Table 1, $\beta_u \approx 1$ occurs at $\varepsilon = 0.371$. Therefore, the integration coefficients β_u, β_{uu}, β_{uuu} and β_{ww} is the value at $\varepsilon = 0.371$. Similarly, the transformation coefficients presented in Table 2 are also evaluated at $\varepsilon = 0.371$.

Table (2). The values of transformation coefficient

ε	$\alpha_{u\eta}$	$\alpha_{w\eta}$
0.37	3.14352	3.25372
0.371	3.15303	3.26436
0.372	3.16258	3.27503
0.373	3.17215	3.28575
0.374	3.18175	3.29650
0.375	3.19137	3.30729
0.376	3.20103	3.31811
0.377	3.21071	3.32898
0.378	3.22041	3.33988
0.379	3.23014	3.35082

III. INTEGRAL SOLUTION USING THE VELOCITY POTENTIAL EQUATION

In the formulation of the velocity equations in Section V, an integration appears that can be solved by applying the velocity potential. This section presents the procedure for evaluating the integral. The integral to be solved is $\frac{\partial}{\partial x}\int_z^{\eta}\left(\frac{\partial}{\partial t}\int_z^{\eta}\frac{\partial u}{\partial x}\,dz\right)dz$.

From (2), the following is obtained

$$\frac{\partial u}{\partial x} = 2Gk^2\cos kx\cosh k(h+z)\sin\sigma t$$

$$\int_z^{\eta}\frac{\partial u}{\partial x}\,dz = 2Gk\cos kx$$

$$(\sinh k(h+\eta) - \sinh k(h+z))\sin\sigma t$$

Considering (4), hence

$$\int_z^{\eta}\frac{\partial u}{\partial x}\,dz = -w_{\eta} + w$$

therefore,

$$\frac{\partial}{\partial t}\int_z^{\eta}\frac{\partial u}{\partial x}\,dz = -\frac{\partial w_{\eta}}{\partial t} + \frac{\partial w}{\partial t}$$

Integrating with respect to

$$\int_z^{\eta}\left(\frac{\partial}{\partial t}\int_z^{\eta}\frac{\partial u}{\partial x}\,dz\right)dz = -\frac{\partial w_{\eta}}{\partial t}(\eta - z) + \int_z^{\eta}\frac{\partial w}{\partial t}dz$$

$$\frac{\partial}{\partial x}\int_z^{\eta}\left(\frac{\partial}{\partial t}\int_z^{\eta}\frac{\partial u}{\partial x}\,dz\right)dz = -\frac{\partial w_{\eta}}{\partial t}\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x}\int_z^{\eta}\frac{\partial w}{\partial t}dz$$

Using (4),

$$\int_z^{\eta}\frac{\partial w}{\partial t}\,dz = -2G\sigma\cos kx$$

$$(\cosh k(h+\eta) - \cosh k(h+z))\cos\sigma t$$

Therefore

$$\frac{\partial}{\partial x}\int_z^{\eta}\frac{\partial w}{\partial t}\,dz = 2Gk\sigma\sin kx$$

$$(\cosh k(h+\eta) - \cosh k(h+z))\cos\sigma t$$

From (2), obtains

$$\frac{\partial}{\partial x}\int_z^{\eta}\frac{\partial w}{\partial t}\,dz = \left(\frac{\partial u_{\eta}}{\partial t} - \frac{\partial u}{\partial t}\right)$$

The final result is,

$$\frac{\partial}{\partial x}\int_z^{\eta}\left(\frac{\partial}{\partial t}\int_z^{\eta}\frac{\partial u}{\partial x}\,dz\right)dz =$$

$$-\frac{\partial w_{\eta}}{\partial t}\frac{\partial \eta}{\partial x} + \left(\frac{\partial u_{\eta}}{\partial t} - \frac{\partial u}{\partial t}\right) \quad \dots(18)$$

IV. WATER SURFACE ELEVATION EQUATION

The water surface elevation equation is formulated using the continuity equation and the principle of energy conservation.

a. Continuity Equation

The continuity equation, derived from the principle of mass conservation in the (x, z) plane is expressed as,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad \dots(19)$$

Integrating this equation along the vertical axis$-z$ from seabed $z = -h$, to water surface $z = \eta$ and substituting the weighted Kinematic Free Surface Boundary Condition, yields,

$$\int_{-h}^{\eta}\frac{\partial u}{\partial x}\,dz + w_{\eta} - w_{-h} = 0$$

Weighted Kinematic Free Surface Boundary Condition is,

$$w_{\eta} = \gamma_{t,2}\frac{\partial \eta}{\partial t} + \gamma_{x,2}u_{\eta}\frac{\partial \eta}{\partial x} \quad \dots(20)$$

where $\gamma_{t,2}$ and $\gamma_{x,2}$ is the weighting coefficients for funcion $f = f(x,t)$. In this research, the values $\gamma_{t,2} = 1.9973$ and $\gamma_{x,2} = 0.9973$ is used. The computational method for these coefficients can be found in Hutahaean (2025b).

$$\int_{-h}^{\eta}\frac{\partial u}{\partial x}\,dz + \gamma_{t,2}\frac{\partial \eta}{\partial t} + \gamma_{x,2}u_{\eta}\frac{\partial \eta}{\partial x} - w_{-h} = 0$$

This equation is subsequently expressed as the water surface elevation equation.,

$$\gamma_{t,2}\frac{\partial \eta}{\partial t} = -\int_{-h}^{\eta}\frac{\partial u}{\partial x}\,dz - \gamma_{x,2}u_{\eta}\frac{\partial \eta}{\partial x} + w_{-h}$$

The integration of the first term on the right-hand side is evaluated using Leibniz's rule of integration (Protter, Murray, Morrey, & Charles, 1985),

$$\int_{\alpha}^{\beta}\frac{\partial f}{\partial x}dz = \frac{\partial}{\partial x}\int_{\alpha}^{\beta}u\,dz - f_{\beta}\frac{\partial \beta}{\partial x} + f_{\alpha}\frac{\partial \alpha}{\partial x} \quad \dots(21)$$

Thus,

$$\int_{-h}^{\eta}\frac{\partial u}{\partial x}dz = \frac{\partial}{\partial x}\int_{-h}^{\eta}u\,dz - u_{\eta}\frac{\partial \eta}{\partial x} - u_{-h}\frac{\partial h}{\partial x}$$

Accordingly,

$$\gamma_{t,2}\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}\int_{-h}^{\eta}u\,dz + u_{\eta}\frac{\partial \eta}{\partial x}$$

$$+ u_{-h}\frac{\partial h}{\partial x} - \gamma_{x,2}u_{\eta}\frac{\partial \eta}{\partial x} + w_{-h}$$

Applying the seabed kinematic boundary condition, $w_{-h} = -u_{-h}\frac{\partial h}{\partial x}$ the third and fifth terms on the right-hand side cancel out. The first term is then expressed using the concept of depth-averaged velocity (6), while the surface horizontal velocity u_η is transformed into the depth-averaged velocity U.

$$\gamma_{t,2}\frac{\partial\eta}{\partial t}=-\frac{\partial\beta_u UD}{\partial x}+\left(1-\gamma_{x,2}\right)\alpha_{u\eta}U\frac{\partial\eta}{\partial x}\quad ...(22)$$

This equation represents the water surface elevation equation derived from the continuity principle. In this formulation, the left-hand side expresses the change in potential energy, yet no source of energy appears on the right-hand side. The missing source term should correspond to kinetic energy. Therefore, this equation must be superimposed with the kinetic energy conservation equation.

b. Energy Conservation Equation

The kinetic energy conservation equation as formulated by Hutahaean (2024b) is expressed as follows,

$$\frac{\partial E_{kx}}{\partial t}+\frac{\partial E_{kz}}{\partial t}=-\frac{\partial uE_{kx}}{\partial x}-\frac{\partial wE_{kz}}{\partial z}\quad ..(23)$$

$E_{kx}=\frac{uu}{2g}$, represents the kinetic energy of the horizontal velocity component, and $E_{kz}=\frac{ww}{2g}$, represents the kinetic energy of the vertical velocity component. Equation (23) is then multiplied by dz and integrated with respect to the vertical axis-z,

$$\int_{-h}^{\eta}\frac{\partial E_{kx}}{\partial t}dz+\int_{-h}^{\eta}\frac{\partial E_{kz}}{\partial t}dz=-\int_{-h}^{\eta}\frac{\partial uE_{kx}}{\partial x}dz-\frac{w_\eta w_\eta w_\eta}{2g}\quad ..(24)$$

Where $\frac{w_{-h}w_{-h}w_{-h}}{2g}$ is neglected. The integration is solved using the Leibniz integration rule,

$$\int_{-h}^{\eta}\frac{\partial E_{kx}}{\partial t}dz=\frac{1}{2g}\frac{\partial}{\partial t}\int_{-h}^{\eta}uu\,dz-\frac{u_\eta u_\eta}{2g}\frac{\partial\eta}{\partial t}$$

The right-hand side is evaluated using the concept of depth-averaged velocity,

$$\int_{-h}^{\eta}\frac{\partial E_{kx}}{\partial t}dz=\frac{1}{2g}\frac{\partial\beta_{uu}UUD}{\partial t}-\frac{u_\eta u_\eta}{2g}\frac{\partial\eta}{\partial t}$$

Considering that $D=h+\eta$ and $u_\eta=\alpha_{u\eta}U$,

$$\int_{-h}^{\eta}\frac{\partial E_{kx}}{\partial t}dz=\frac{\beta_{uu}D}{2g}\frac{\partial UU}{\partial t}+\left(\beta_{uu}-\alpha_{u\eta}\alpha_{u\eta}\right)\frac{UU}{2g}\frac{\partial\eta}{\partial t}$$

In a similar way, the following is obtained:

$$\int_{-h}^{\eta}\frac{\partial E_{kz}}{\partial t}dz=\frac{\beta_{ww}D}{2g}\frac{\partial WW}{\partial t}+\left(\beta_{ww}-\alpha_{w\eta}\alpha_{w\eta}\right)\frac{WW}{2g}\frac{\partial\eta}{\partial t}$$

By substituting the results of these integrations into Equation (23), we obtain the following expression:

$$\lambda\frac{\partial\eta}{\partial t}=-\frac{1}{2g}\frac{\partial UU}{\partial t}-\frac{\beta_{ww}}{2g\beta_{uu}}\frac{\partial WW}{\partial t}-\frac{1}{2g\beta_{uu}D}\int_{-h}^{\eta}\frac{\partial uE_{kx}}{\partial x}dz-\frac{1}{2g\beta_{uu}D}w_\eta w_\eta w_\eta\quad ...(25)$$

Where,

$$\lambda=\frac{\left(\left(\beta_{uu}-\alpha_{u\eta}\alpha_{u\eta}\right)UU+\left(\beta_{ww}-\alpha_{w\eta}\alpha_{w\eta}\right)WW\right)}{2g\beta_u D}$$

The term g on both the left-hand side and the right-hand side cannot be eliminated, as doing so would result in inconsistent dimensional units for $\frac{\partial\eta}{\partial t}$ and cannot be undergo superposition with $\frac{\partial\eta}{\partial t}$ within the continuity equation. The integration of the third term on the right-hand side can be solved using Leibniz integration and the concept of depth-averaged velocity,

$$\frac{1}{2g}\int_{-h}^{\eta}\frac{\partial uuu}{\partial x}dz=\frac{\beta_{uuu}}{2g}\frac{\partial UUUD}{\partial x}-\frac{u_\eta u_\eta u_\eta}{2g}\frac{\partial\eta}{\partial x}-u_{-h}u_{-h}u_{-h}\frac{\partial h}{\partial x}$$

By neglecting the term $u_{-h}u_{-h}u_{-h}\frac{\partial h}{\partial x}$ and transforming $\frac{u_\eta u_\eta u_\eta}{2g}=\frac{\alpha_{u\eta}\alpha_{u\eta}\alpha_{u\eta}UUU}{2g}$.

$$\frac{1}{2g}\int_{-h}^{\eta}\frac{\partial uuu}{\partial x}dz=\frac{\beta_{uuu}}{2g}\frac{\partial UUUD}{\partial x}-\frac{\alpha_{u\eta}\alpha_{u\eta}\alpha_{u\eta}}{2g}UUU\frac{\partial\eta}{\partial x}$$

Subsequently, Equation (22) is superposed with Equation (25), yielding:

$$\gamma_{t,2}\frac{\partial\eta}{\partial t}=-\frac{\partial\beta_u UD}{\partial x}+\left(1-\gamma_{x,2}\right)\alpha_{u\eta}U\frac{\partial\eta}{\partial x}$$

$$\lambda\frac{\partial\eta}{\partial t}=-\frac{1}{2g}\frac{\partial UU}{\partial t}-\frac{\beta_{ww}}{2g\beta_{uu}}\frac{\partial WW}{\partial t}-\frac{1}{2g\beta_{uu}D}\int_{-h}^{\eta}\frac{\partial uE_{kx}}{\partial x}dz-\frac{1}{2g\beta_{uu}D}w_\eta w_\eta w_\eta$$

$$\left(\gamma_{t,2}+\lambda\right)\frac{\partial\eta}{\partial t}=-\frac{\partial\beta_u UD}{\partial x}+\left(1-\gamma_{x,2}\right)\alpha_{u\eta}U\frac{\partial\eta}{\partial x}-\frac{1}{2g}\frac{\partial UU}{\partial t}-\frac{\beta_{ww}}{2g\beta_{uu}}\frac{\partial WW}{\partial t}-\frac{1}{2g\beta_{uu}D}\int_{-h}^{\eta}\frac{\partial uE_{kx}}{\partial x}dz-\frac{1}{2g\beta_{uu}D}w_\eta w_\eta w_\eta\quad ...(26)$$

Equation (26) represents the water surface elevation equation, which is used to calculate $\eta(x,t)$.

V. HORIZONTAL WATER PARTICLE VELOCITY EQUATION

The horizontal water particle velocity equation is derived using the Euler momentum conservation equations.

5.1. Euler Momentum Conservation Equations.

The Euler momentum conservation equations for flow in the (x, z) plane consist of two parts: the particle velocity equation in the horizontal direction and the particle velocity equation in the vertical direction. By employing the concept of weighted total acceleration, these equations are expressed as follows:

Horizontal particle velocity equation,

$$\gamma_{t,3}\frac{\partial u}{\partial t}+\gamma_{x,3}\,u\frac{\partial u}{\partial x}+\gamma_{z,3}\,w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad .(27)$$

Vertical particle velocity equation,

$$\gamma_{t,3}\frac{\partial w}{\partial t}+\gamma_{x,3}u\frac{\partial w}{\partial x}+\gamma_{z,3}\,w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g \qquad ..(28)$$

ρ is the water density, p s the pressure acting on the fluid particle, and g is the gravitational acceleration.

Equations (27) and (28) are formulated under the condition that the horizontal particle velocity u varies with respect to both the horizontal axis -x and the vertical axis -z, while the vertical particle velocity w similarly varies with respect to both axes. In this research, however, the Euler momentum conservation equations are developed under the assumption that the horizontal velocity u varies only along the horizontal axis -x and the vertical velocity w varies only along the vertical axis -z. This assumption ensures consistency with the formulation of the continuity equation as follows.

$$\gamma_{t,3}\frac{\partial u}{\partial t}+\frac{\gamma_{x,3}}{2}\frac{\partial uu}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ....(29)$$

$$\gamma_{t,3}\frac{\partial w}{\partial t}+\frac{\gamma_{z,3}}{2}\frac{\partial ww}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g \qquad ...(30)$$

$\gamma_{t,3}$, $\gamma_{x,3}$ and $\gamma_{z,3}$ are the weighting coefficients in the weighted Taylor series. This research employed $\gamma_{t,3}=3.04933$ and $\gamma_{x,3}=\gamma_{z,3}=2.04933$, which calculation method of the weighting coefficients referred to Hutahaean (2025b).

The horizontal water particle velocity equation is formulated using Equations (29) and (30), with the pressure equation p derived first. To ensure a strong interaction between the velocity equation and the continuity equation, the formulation of pressure p is based on the continuity equation. Integrating the continuity equation along the vertical axis-z, the following is obtained

$$\int_z^\eta \frac{\partial u}{\partial x}dz+w_\eta-w=0$$

Differentiating this equation with respect to time-t yields an expression for $\frac{\partial w}{\partial t}$,

$$\frac{\partial w}{\partial t}=\frac{\partial}{\partial t}\int_z^\eta \frac{\partial u}{\partial x}dz+\frac{\partial w_\eta}{\partial t}$$

Substituting to (30),

$$\gamma_{t,3}\left(\frac{\partial}{\partial t}\int_z^\eta \frac{\partial u}{\partial x}dz+\frac{\partial w_\eta}{\partial t}\right)+\frac{\gamma_{z,3}}{2}\frac{\partial ww}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g$$

This expression is then rearranged into an equation for pressure p , which is integrated over the water depth from $z=z$ to $z=\eta$ applying the dynamic surface boundary condition $p_\eta=0$. This yields the following pressure equation p.

$$\frac{p}{\rho}=\gamma_{t,3}\int_z^\eta\left(\frac{\partial}{\partial t}\int_z^\eta \frac{\partial u}{\partial x}dz\right)dz+\frac{\gamma_{z,3}}{2}\left(w_\eta w_\eta-ww\right)+\left(g+\gamma_{t,3}\frac{\partial w_\eta}{\partial t}\right)(\eta-z)$$

Differentiating with respect to the horizontal axis-x, while neglecting the cross-differential term, $\frac{\partial}{\partial x}\left(\frac{\partial w_\eta}{\partial t}\right)$,

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=\gamma_{t,3}\frac{\partial}{\partial x}\int_z^\eta\left(\frac{\partial}{\partial t}\int_z^\eta \frac{\partial u}{\partial x}dz\right)dz+\frac{\gamma_{z,3}}{2}\frac{\partial}{\partial x}\left(w_\eta w_\eta-ww\right)+\left(g+\gamma_{t,3}\frac{\partial w_\eta}{\partial t}\right)\frac{\partial \eta}{\partial x} \qquad ..(31)$$

5.2. Formulation Using the Direct Method

Equation (31) is substituted into Equation (29) and evaluated at $z=\eta$ known as the direct method.

$$\gamma_{t,3}\frac{\partial u_\eta}{\partial t}+\frac{\gamma_{x,3}}{2}\frac{\partial u_\eta u_\eta}{\partial x}=-\left(g+\gamma_{t,3}\frac{\partial w_\eta}{\partial t}\right)\frac{\partial \eta}{\partial x}$$

The second term on the left-hand side is assigned a negative sign, based on the assumption that it represents the hydrodynamic force acting in the direction of energy transfer from higher to lower energy states (Hutahaean, 2025a). The equation can therefore be rearranged as:

$$\gamma_{t,3}\frac{\partial u_\eta}{\partial t}=\frac{\gamma_{x,3}}{2}\frac{\partial u_\eta u_\eta}{\partial x}-\left(g+\gamma_{t,3}\frac{\partial w_\eta}{\partial t}\right)\frac{\partial \eta}{\partial x}$$

Next, the variables on the left-hand side and the first term on the right-hand side are transformed into depth-averaged velocity form,

$$\gamma_{t,3}\alpha_{u\eta}\frac{\partial U}{\partial t}=\frac{\gamma_{x,3}\alpha_{u\eta}\alpha_{u\eta}}{2}\frac{\partial UU}{\partial x}-\left(g+\gamma_{t,3}\frac{\partial w_\eta}{\partial t}\right)\frac{\partial \eta}{\partial x} \qquad ...(32)$$

The term $\frac{\partial w_\eta}{\partial t}$ on the right-hand side is expressed using the continuity equation. Integrating the continuity equation

(19) along the vertical axis-z, from $z=-h$ to $z=\eta$, applying the Leibniz integration rule, the depth-averaged velocity concept, and the seabed kinematic boundary condition, yields,

$$w_\eta = -\frac{\partial \beta_u UD}{\partial x} + u_\eta \frac{\partial \eta}{\partial x}$$

By neglecting second-order differentials and cross-differentials, the following relation is obtained:

$$\frac{\partial w_\eta}{\partial t} = -\beta_u \frac{\partial U}{\partial x}\frac{\partial \eta}{\partial t} + \left(\alpha_{u\eta}\frac{\partial \eta}{\partial x} - \beta_u \frac{\partial D}{\partial x}\right)\frac{\partial U}{\partial t} \quad \text{... (33)}$$

Substituting to (32),

$$\gamma_{t,3}\left(\alpha_{u\eta} + \left(\alpha_{u\eta}\frac{\partial \eta}{\partial x} - \beta_u\frac{\partial D}{\partial x}\right)\frac{\partial \eta}{\partial x}\right)\frac{\partial U}{\partial t} = \frac{\gamma_{x,3}\alpha_{u\eta}\alpha_{u\eta}}{2}\frac{\partial UU}{\partial x} - \left(g - \gamma_{t,3}\beta_u\frac{\partial U}{\partial x}\frac{\partial \eta}{\partial t}\right)\frac{\partial \eta}{\partial x} \quad \text{... (34)}$$

In the second term on the right-hand side, the parameter $\frac{\partial \eta}{\partial t}$ appears, which represents the source of energy contributing to changes in kinetic energy on the left-hand side.

This equation does not contain the variable water depth. As a result, the shoaling-breaking modeling based on this formulation produces large breaking wave heights, occurring only in very shallow waters.

For this reason, a subsequent model was developed using the integration method, which incorporates the water depth variable into the formulation.

5.3. Formulation with the Integration Method

In this section, the formulation of the velocity equation is carried out through integration, hence it is referred to as the integration method. The integration of the first term on the right-hand side of Equation (31) is performed using the velocity potential equation (1), yielding Equation (18). Substituting Equation (18) into (31) gives:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \gamma_{t,3}\left(\frac{\partial u_\eta}{\partial t} - \frac{\partial u}{\partial t}\right) + \frac{\gamma_{z,3}}{2}\frac{\partial w_\eta w_\eta}{\partial x} - \frac{\gamma_{z,3}}{2}\frac{\partial ww}{\partial x} + g\frac{\partial \eta}{\partial x}$$

Substituting (29),

$$\gamma_{t,3}\frac{\partial u_\eta}{\partial t} + \frac{\gamma_{x,3}}{2}\frac{\partial uu}{\partial x} = -\frac{\gamma_{z,3}}{2}\frac{\partial w_\eta w_\eta}{\partial x} + \frac{\gamma_{z,3}}{2}\frac{\partial ww}{\partial x} - g\frac{\partial \eta}{\partial x}$$

The second term on the left-hand side, as well as the first and second terms on the right-hand side, represent hydrodynamic forces directed positively from higher to lower energy. Therefore, following Hutahaean (2025b),

$$\gamma_{t,3}\frac{\partial u_\eta}{\partial t} - \frac{\gamma_{x,3}}{2}\frac{\partial uu}{\partial x} = \frac{\gamma_{z,3}}{2}\frac{\partial w_\eta w_\eta}{\partial x} - \frac{\gamma_{z,3}}{2}\frac{\partial ww}{\partial x} - g\frac{\partial \eta}{\partial x}$$

Integrating with respect to the vertical axis-z,

$$\gamma_{t,3}D\frac{\partial u_\eta}{\partial t} - \frac{\gamma_{x,3}}{2}\int_{-h}^{\eta}\frac{\partial uu}{\partial x}dz = \frac{\gamma_{z,3}D}{2}\frac{\partial w_\eta w_\eta}{\partial x} - \frac{\gamma_{z,3}}{2}\int_{-h}^{\eta}\frac{\partial ww}{\partial x}dz - gD\frac{\partial \eta}{\partial x}$$

Dividing through by D, and transforming the first term on the left-hand side into the depth-averaged velocity, gives:

$$\gamma_{t,3}\alpha_{u\eta}\frac{\partial U}{\partial t} = \frac{\gamma_{x,3}}{2D}\int_{-h}^{\eta}\frac{\partial uu}{\partial x}dz + \frac{\gamma_{z,3}}{2}\frac{\partial w_\eta w_\eta}{\partial x} - \frac{\gamma_{z,3}}{2D}\int_{-h}^{\eta}\frac{\partial ww}{\partial x}dz - g\frac{\partial \eta}{\partial x} \quad \text{... (35)}$$

In this equation, the parameter $\frac{\partial \eta}{\partial t}$ does not appear explicitly; however, it is implicitly contained in the term w_η (second term on the right-hand side) and in w (third term on the right-hand side).

Equation (35) thus serves as the governing relation for computing the depth-averaged horizontal velocity. By neglecting seabed velocity, the integrals are expressed as,

$$\int_{-h}^{\eta}\frac{\partial uu}{\partial x}dz = \beta_{uu}\frac{\partial UUD}{\partial x} - u_\eta u_\eta\frac{\partial \eta}{\partial x}$$

$$\int_{-h}^{\eta}\frac{\partial ww}{\partial x}dz = \beta_{ww}\frac{\partial WWD}{\partial x} - w_\eta w_\eta\frac{\partial \eta}{\partial x}$$

The shoaling-breaking modeling based on the velocity equation (35) predicts that waves undergo breaking more readily, with breaking occurring at water depths that are still relatively large.

5.4. Final Form of the Horizontal Velocity Equation

The final form of the horizontal water particle velocity equation is obtained by combining equations (34) and (35), with the composition defined as α_1x (34) + α_2 x ((35), where $\alpha_1 + \alpha_2 = 1.0$.

$$\alpha_1\gamma_{t,3}\left(\alpha_{u\eta} + \left(\alpha_{u\eta}\frac{\partial \eta}{\partial x} - \beta_u\frac{\partial D}{\partial x}\right)\frac{\partial \eta}{\partial x}\right)\frac{\partial U}{\partial t} = \alpha_1\left(\frac{\gamma_{x,3}\alpha_{u\eta}\alpha_{u\eta}}{2}\frac{\partial UU}{\partial x} - \left(g - \gamma_{t,3}\beta_u\frac{\partial U}{\partial x}\frac{\partial \eta}{\partial t}\right)\frac{\partial \eta}{\partial x}\right)$$

$$\alpha_2\gamma_{t,3}\alpha_{u\eta}\frac{\partial U}{\partial t} = \alpha_2\left(\frac{\gamma_{x,3}}{2D}\int_{-h}^{\eta}\frac{\partial uu}{\partial x}dz + \frac{\gamma_{z,3}}{2}\frac{\partial w_\eta w_\eta}{\partial x} - \frac{\gamma_{z,3}}{2D}\int_{-h}^{\eta}\frac{\partial ww}{\partial x}dz - g\frac{\partial \eta}{\partial x}\right)$$

The two equations were summed,

$$\gamma_{t,3}\left(\left(\alpha_1+\alpha_2\right)\alpha_{u\eta}+\alpha_1\left(\alpha_{u\eta}\frac{\partial \eta}{\partial x}-\beta_u\frac{\partial D}{\partial x}\right)\frac{\partial \eta}{\partial x}\right)\frac{\partial U}{\partial t}$$

$$=\alpha_1\left(\frac{\gamma_{x,3}\alpha_{u\eta}\alpha_{u\eta}}{2}\frac{\partial UU}{\partial x}-\left(g-\gamma_{t,3}\beta_u\frac{\partial U}{\partial x}\frac{\partial \eta}{\partial t}\right)\frac{\partial \eta}{\partial x}\right)$$

$$+\alpha_2\left(\frac{\gamma_{x,3}}{2D}\int_{-h}^{\eta}\frac{\partial uu}{\partial x}dz+\frac{\gamma_{z,3}}{2}\frac{\partial w_\eta w_\eta}{\partial x}\right)$$

$$-\left(\frac{\gamma_{z,3}}{2D}\int_{-h}^{\eta}\frac{\partial ww}{\partial x}dz-g\frac{\partial \eta}{\partial x}\right) \quad ...(36)$$

Equation (36) represents the final equation for calculating the horizontal depth-averaged velocity. In this research, the weighting coefficients were chosen as $\alpha_1 = 0.4$ and $\alpha_2 = 0.6$, in order to prevent the breaking wave height from becoming excessively large and to avoid overly deep breaking depths.

5.5 Differential Calculation of Vertical Velocity w.

Equation (36) contains a differential term involving vertical velocity. In this research, the differential of the vertical velocity was calculated as follows.

By integrating the continuity equation, the vertical velocity expression is obtained:

$$w_\eta=-\frac{\partial \beta_u UD}{\partial x}+u_\eta\frac{\partial \eta}{\partial x}$$

Neglecting cross-differentials and second-order differentials,

$$\frac{\partial w_\eta}{\partial x}=-\beta_u\frac{\partial D}{\partial x}\frac{\partial U}{\partial x}-\beta_u\frac{\partial U}{\partial x}\frac{\partial h}{\partial x}+\left(\alpha_{u\eta}-\beta_u\right)\frac{\partial U}{\partial x}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial w_\eta w_\eta}{\partial x}=2w_\eta\frac{\partial w_\eta}{\partial x}$$

$$\frac{\partial w_\eta w_\eta}{\partial x}=2w_\eta$$

$$\left(-\beta_u\frac{\partial D}{\partial x}\frac{\partial U}{\partial x}-\beta_u\frac{\partial U}{\partial x}\frac{\partial h}{\partial x}+\left(\alpha_{u\eta}-\beta_u\right)\frac{\partial U}{\partial x}\frac{\partial \eta}{\partial x}\right)$$

Thus, the differential of the depth-averaged vertical velocity with respect to the horizontal axis-x,

$$\frac{\partial WW}{\partial x}=\frac{2w_\eta}{\alpha_{w\eta}\alpha_{w\eta}}\left(-\beta_u\frac{\partial D}{\partial x}\frac{\partial U}{\partial x}-\beta_u\frac{\partial U}{\partial x}\frac{\partial h}{\partial x}+\left(\alpha_{u\eta}-\beta_u\right)\frac{\partial U}{\partial x}\frac{\partial \eta}{\partial x}\right) \quad ...(37)$$

VI. NUMERICAL SOLUTION

Equations (25) and (36) were solved numerically using the Finite Difference Method for spatial derivatives and the predictor–corrector method for time derivatives. In the predictor stage, a central difference scheme was employed, while in the corrector stage, numerical integration using the Newton–Cotes method was applied. Detailed steps of the predictor–corrector procedure can be found in Hutahaean (2024a). The selection of time step δt and grid-size δx is accessible in Hutahaean (2025a).

VII. EXAMPLE OF MODEL RESULTS

This section presents several results obtained from model execution.

7.1. Wave Input

The model was provided with a sinusoidal wave input expressed by the equation,

$$\eta(0,t)=-A\cos\sigma t+A \quad ...(38)$$

A is the wave amplitude, $\sigma=\frac{2\pi}{T}$ is the angular frequency and T is the wave-period. This equation corresponds to the initial condition at the starting point $x = 0.0$, at $t = 0$, $\eta(0,0) = 0.0$ and $\frac{\partial \eta}{\partial t} = 0.0$. The input wave curve as a function of time t exhibits a solitary wave form, as shown in Fig. 2, for a wave period of $T = 8.0\ sec.$, and wave amplitude $A = 1.0\ m$.

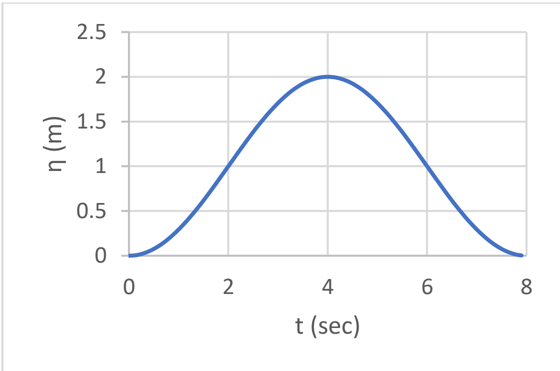

Fig (2) Input wave profile as a function of time -t.

7.2. Model Execution over a Flat Bottom.

The model was executed for five wave periods in water depth h =21.0 m, with a wave period of 8.0 s and wave amplitude 1.2 m where wave height H is 2.4 m as shown in Fig. 2 and Fig. 3.

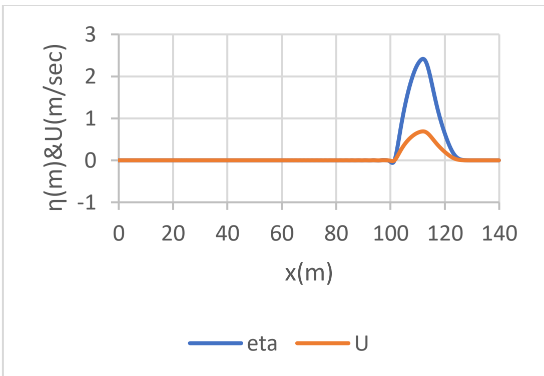

Fig (2). Wave profile after execution of 5 wave periods, $A = 1.2\ m$

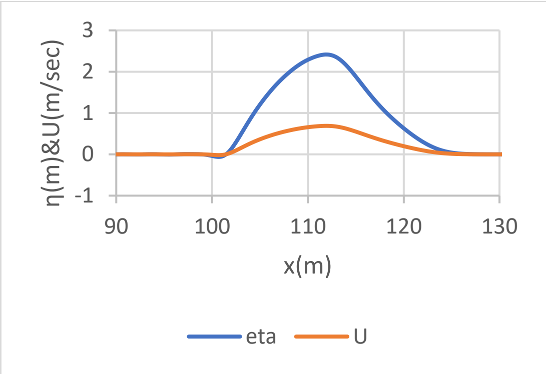

Fig (3). Detailed wave profile, wave amplitude $A = 1.2\ m$

The resulting wave profiles, shown in Fig. 2 and Fig. 3, indicate that the right-hand side of the wave is gentler than the left-hand side. The right-hand portion closely resembles a solitary wave profile, which has been widely used in previous studies. The generated wavelength is 22.0 m, with a wave steepness $\frac{H}{L} = \frac{2.4}{22} = 0.109$, which is still well below the critical wave steepness criteria from Michell (1893) $\left(\frac{H}{L}\right)_{crit} = 0.142$, and Toffoli (2010), $\left(\frac{H}{L}\right)_{crit} = 0.170$. The modeled wavelength is considerably shorter than that predicted by linear wave theory or by the Airy long-wave equation. However, the wave steepness remains well below the critical threshold.

7.3. Model Execution over a Sloping Bottom.

This section presents the results of the model execution over a sloping bottom for five different bottom slopes. The bathymetric cross-section is shown in Fig. 4. The simulated waves have a wave period of 8.0 s and a wave amplitude of 1.20 m.

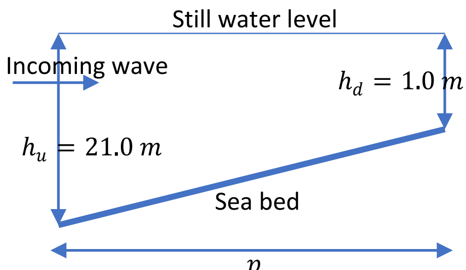

Fig (4). Bathymetric cross-section (non scale).

In Fig. 4, the deep-water depth is $h_u = 21.0\ m$ and shallow water depth $h_d = 1.0\ m$. Five values of horizontal distance p of 400 m, 300 m, 200 m, 100 m and 50 m were used, thereby 5 bottom slopes were obtained, $\frac{20}{400}, \frac{20}{300}, \frac{20}{200}, \frac{20}{100}$ and $\frac{20}{50}$.

a. Model Execution for Bottom Slope $\frac{20}{400} = 0.05$

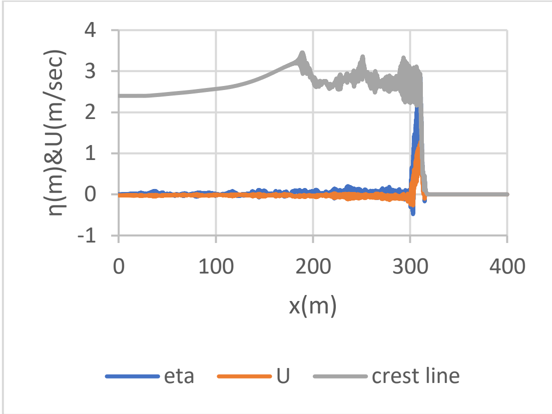

Fig (5). Shoaling and breaking over bottom slope 0.05.

In Fig. 5 and Fig. 6, the crest line represents the line connecting the wave crests. Wave breaking begins at $x = 190.0\ m$, at water depth $h = 11.5\ m$. Breaking at this point can be classified as soft breaking, characterized by the appearance of foam at the wave crest. Breaking then continues until $x = 310.5\ m$ at breaker depth $h_b = 5.48\ m$, where hard breaking occurs and the model terminates.

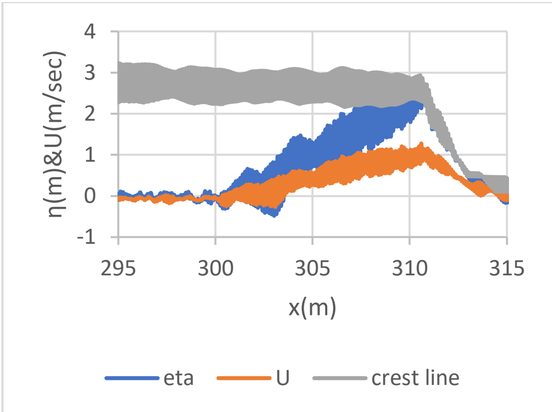

Fig (6). Breaking wave profile over bottom slope 0.05.

The parameters of the hard breaking are as follows:

$h_b = 5.48\ m$, $L_b = 16.0\ m$, $H_b = 2.60\ m$,

$\frac{H_b}{h_b} = 0.475$, $\frac{H_b}{L_b} = 0.163$

In this research, the breaking point is defined as the location where the model execution could no longer continue.

b. Bottom slope $\frac{20}{300} = 0.067$

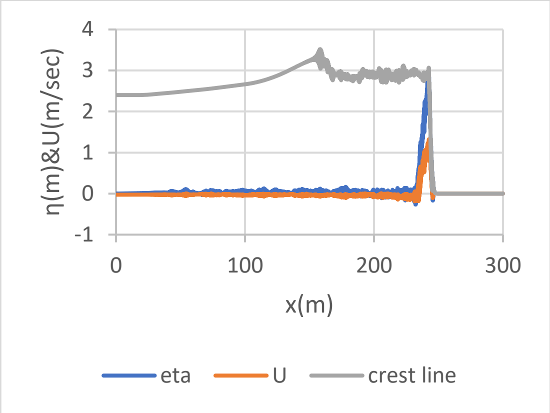

Fig (7). Shoaling breaking over bottom slope 0.067.

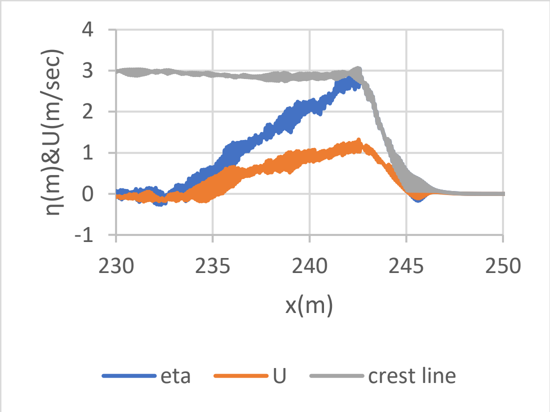

Fig (8). Breaking wave profile at bottom slope 0.067.

Parameter of breaking over bottom slope 0.067 is,

$h_b = 4.83\ m$, $L_b = 13.0\ m$, $H_b = 2.80\ m$,

$\frac{H_b}{h_b} = 0.58$, $\frac{H_b}{L_b} = 0.215$

c. Bottom slope $\frac{20}{200} = 0.10$

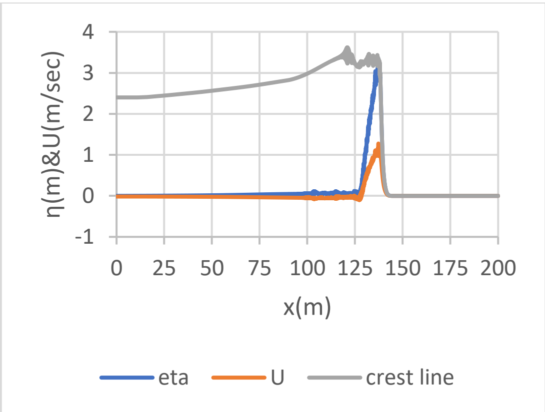

Fig (9). Shoaling breaking over bottom slope 0.10.

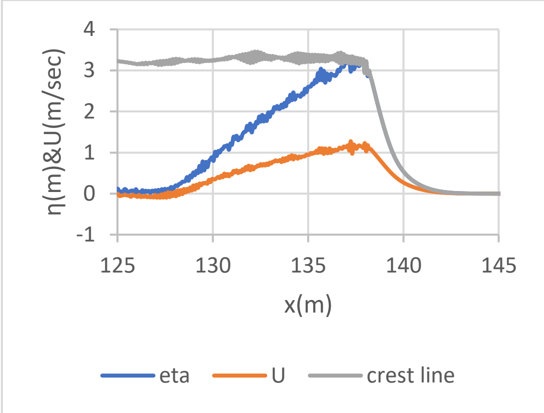

Fig (10). Breaking wave profile over bottom slope 0.1.

Parameter breaking at bottom slope 0.1 is,

$h_b = 7.25\ m$, $L_b = 15.0\ m$, $H_b = 3.20\ m$,

$\frac{H_b}{h_b} = 0.441$, $\frac{H_b}{L_b} = 0.213$

d. Bottom slope $\frac{20}{100} = 0.200$

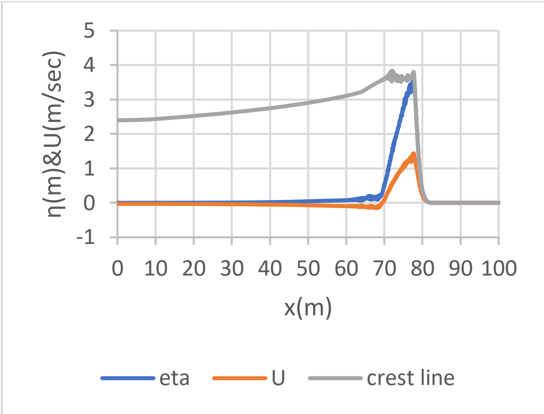

Fig (11). Shoaling breaking over bottom slope 0.200.

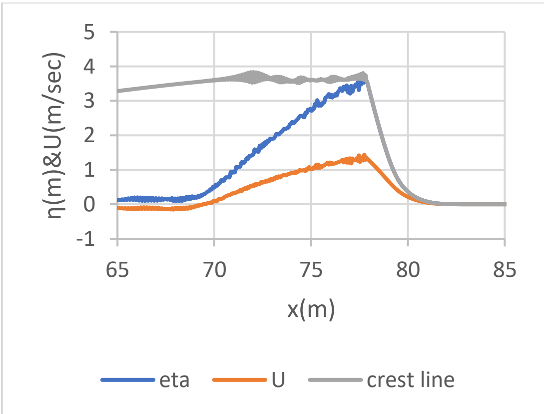

Fig (12). Breaking wave profile over bottom slope 0.200.

The breaking parameter over bottom slope 0.200 is,

$h_b = 5.44\ m$, $L_b = 13.0\ m$, $H_b = 3.60\ m$,

$\frac{H_b}{h_b} = 0.662, \frac{H_b}{L_b} = 0.277$

e. Bottom slope $\frac{20}{50} = 0.4$

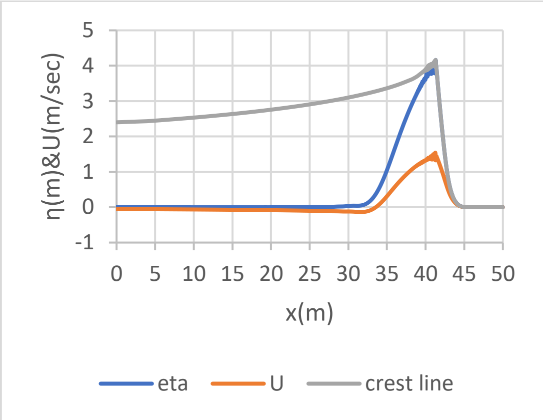

Fig (13). Shoaling breaking over bottom slope 0.4.

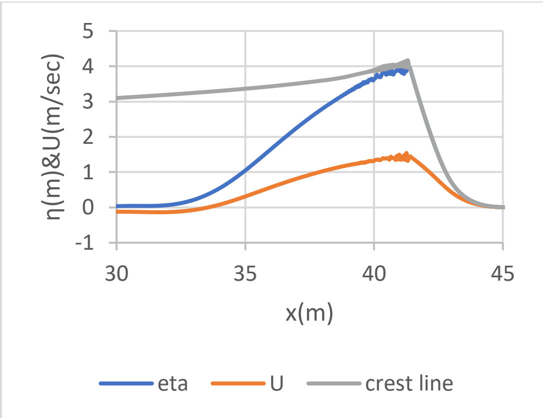

Fig (14). Breaking wave profile over bottom slope 0.4.

Parameter of breaking over bottom slope 0.4 is,

$h_b = 4.4\ m$, $L_b = 12.0\ m$, $H_b = 4.00\ m$,

$\frac{H_b}{h_b} = 0.909$, $\frac{H_b}{L_b} = 0.333$

Table (3). Summary of the breaking parameters

Slope	h_b (m)	L_b (m)	H_b (m)	$\frac{H_b}{h_b}$	$\frac{H_b}{L_b}$
0.050	5.48	16.0	2.6	0.475	0.162
0.067	4.83	13.0	2.8	0.579	0.215
0.100	7.25	15.0	3.2	0.441	0.213
0.200	5.44	13.0	3.6	0.662	0.277
0.400	4.40	12.0	4.0	0.909	0.333

The summary of breaking parameters is presented in Table 3. In general, it can be observed that the steeper the bottom slope, the shallower the breaker depth h_b, although deviations occur for the bottom slope of 0.10. In contrast, the breaking wave height tends to increase with increasing bottom slope.

7.4. Comparison with Previous Studies.

The influence of bottom slope on breaking parameters is well-established. Many researchers have formulated breaking index equations incorporating bottom slope as a parameter.

a.Breaking height index $\frac{H_b}{H_0}$

Several researchers have proposed relationships between the breaking wave height H_b with deep water wave height H_0 using bottom slope as the parameters as follows.

a.1.Le Mehaute and Koh (1967)

$$\frac{H_b}{H_0} = 0.76 m^{1/7} \left(\frac{H_0}{L_0}\right)^{-0.25}$$

m is the bottom slope

a.2.Sunamura and Horikawa (1974),

$$\frac{H_b}{H_0} = m^{0.2} \left(\frac{H_0}{L_0}\right)^{-0.25}$$

a.3. Ogawa and Shuto (1984)

$$\frac{H_b}{H_0} = 0.68\ m^{0.09} \left(\frac{H_0}{L_0}\right)^{-0.25}$$

The comparison between the model results and these previous studies is presented in Table 2.

Table (2) Comparison of $\frac{H_b}{H_0}$

m	$a.1$	$a.2.$	$a.3$	Model
0.05	1.395	1.319	1.258	1.083
0.067	1.478	1.354	1.311	1.167
0.1	1.603	1.404	1.389	1.333
0.2	1.841	1.494	1.534	1.5
0.4	2.115	1.591	1.694	1.667

From Table 2, it can be seen that the model results are generally close to those reported in previous studies, with the largest deviation occurring at a bottom slope of 0.05.

b. Breaking depth index $\frac{H_b}{h_b}$

Several studies have formulated relationships between breaking wave height H_b with breaking water depth h_b using bottom slope as the parameter as follows.

b.1. Collins and Weir (1969)

$$\frac{H_b}{h_b} = 0.72 + 5.6\, m$$

b.2. Sunamura (1980)

$$\frac{H_b}{h_b} = 1.1 \left(\frac{m}{\sqrt{H_0/L_0}} \right)^{1/6}$$

b.3. Larson and Kraus (1989)

$$\frac{H_b}{h_b} = 1.14 \left(\frac{m}{\sqrt{H_0/L_0}} \right)^{0.21}$$

Table (3) Comparison of values $\frac{H_b}{h_b}$

m	$b.1$	$b.2$	$b.3$	Model
0.05	0.748	0.911	0.899	0.474
0.067	0.757	0.956	0.955	0.58
0.1	0.776	1.023	1.04	0.441
0.2	0.832	1.148	1.203	0.662
0.4	0.944	1.288	1.391	0.909

From Table 3, it is evident that there are significant differences between the model results and previous studies. The common trend, however, is that the breaking depth index $\frac{H_b}{h_b}$ increases with increasing bottom slope.

c.Breaking length index $\frac{H_b}{L_b}$

Several studies have proposed relationships between breaking wave height H_b and breaking wave length L_b using bottom slope as a parameter, including:,

c.1. Ostendorf and Madsen (1979)

for $m \leq 0.1$

$$\frac{H_b}{L_b} = 0.14\, tanh \left[(0.8 + 5\, m) \frac{2\pi h_b}{L_0} \right]$$

for $m > 0.1$

$$\frac{H_b}{L_b} = 0.14\, tanh \left[\left(0.8 + 5\,(0.1)\right) \frac{2\pi h_b}{L_0} \right]$$

c.2. Rattanapitikon and Shibayama (2000)

$$\frac{H_b}{L_b} = 0.14\, tanh \left[(-11.21\, m^2 + 5.01\, m + 0.91) \frac{2\pi h_b}{L_0} \right]$$

In both equations, the parameters H_b and breaking wave depth h_b, using H_b and h_b of the model, $\frac{H_b}{L_b}$ is calculated.

Table (4). Comparison of breaking wavelength $\frac{H_b}{L_b}$.

m	c.1	c.2	model
0.05	0.179	0.137	0.162
0.067	0.174	0.133	0.215
0.1	0.179	0.14	0.22
0.2	0.167	0.135	0.277
0.4	0.131	0.068	0.333

The model's breaking wavelength cannot be directly compared with these two equations, as both are based on the critical wave steepness defined by Michell (1893) $\left(\frac{H}{L}\right)_{crit} = 0.142$. Moreover, both equations are derived from linear wave theory, which predicts wavelengths that are much longer than those observed in the model.

VIII. CONCLUSION

This research demonstrates that the developed governing equations, the water surface elevation equation and the horizontal water particle velocity equation, effectively model wave shoaling and breaking processes. The results indicate that both equations capture the essential dynamics of wave transformation, including the evolution of wave height, depth, and horizontal particle motion.

The use of a solitary wave input profile ensures consistency with the initial conditions at the input point, allowing the Kinematic Free Surface Boundary Condition to be applied without difficulty. This enables accurate computation of vertical water velocity and its derivatives across all spatial points. Vertical velocity is a key factor in determining horizontal water particle velocity, as it provides the necessary energy for horizontal motion. For further refinement, a mechanism for wave energy dissipation during hard breaking should be incorporated into both the water surface elevation equation and the horizontal velocity equation.

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