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Boundary value problems of fractional differential equations with nonlocal and integral boundary conditions

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***Abstract*— *The boundary value problems of fractional differential equations involving Caputo derivatives are examined in this article. The ultimate goal of our study is to institute the existence and uniqueness properties for our boundary problems with nonlocal and integral boundary conditions by applying the Banach and the Schaefer's theorems of fixed point. We finally discuss two applicable questions to enhance the comprehension of our outcomes and conclude by summarizing our results and giving vital suggestions for further research works in relation to our study.*

I. INTRODUCTION

The concept of fractional differentials and integrals date as far back as the middle of the nineteenth century by the works of some mathematicians such as Liouville, L'Hospital and Riemann. A century later, engineers and physicists found considerable applications to the concept in their respective fields of interest such as physics, identification of signals and processing of images, aerodynamics, blood flow phenomena and many other relevant fields of study[11,13]. Over the years, numerous concepts of fractional derivatives have been established such as the Riemann-Liouville and the Caputo derivative coupled with an extensive and significant advancements in the analysis of the solutions of FDEs, specifically of higher-order differentials with boundary restrictions.

The importance and applicability of fractional differential equations keep increasing extensively over the years due to its accuracy and objectiveness in describing nonlinear and or natural phenomenon such as stochastic and diffusion models, hydrology processes and finance. Lately, the advancement of fractional derivatives has not only portrayed in-depth research backgrounds but also a vast application to real life situations. The analysis of the multiplicity, uniqueness and existence of solutions and positive solutions of boundary value problems via the application of nonlinear techniques such as the theory of fixed points, the Leray-Schauder theorem, the technique of Upper and Lower solution among others has garnered much attention from researchers[3-13].

Benchohra et al[5], investigated the boundary problems of the FDE below;

$$D_{0^+}^{\propto} y(t) = f\big(t, y(t)\big), \quad t \in J = [0, T], \quad \propto \in (1, 2].$$

$$y(0) = g(y), \qquad y(T) = y_T$$

where $D_{0^+}^{\propto}$ is a Caputo's differential, $f: [0, T] \times R \to R$ and $g: c(J, R) \to R$ are continuous functions and $y_T \in R$.

Cabada and Wang [6], examined the positive solutions of the FDE below with integral boundary conditions;

$$^{c}D^{\alpha} u(t) + f\big(t, u(t)\big) = 0, \qquad 0 < t < 1$$

$$u(0) = u''(0) = 0, \qquad u(1) = \lambda \int_0^1 u(s) ds$$

where $\alpha \in (2,3), \lambda \in (0,2), {}^cD^{\alpha}$ is a Caputo fractional differential, order α and $f:[0,1]\times[0,\infty]\to[0,\infty)$ is a continuous function.

Yan et al.[18], investigated the boundary value problems with integral conditions for the fractional differential equation;

$${}^cD^{\alpha}_{0+}x(t) = f(t, x(t)), \quad t \in [0,T], \quad n-1 < \propto \le 2,$$

$$x(0) = y(x), \qquad \int_0^T x(t)dt = m$$

$$x(0) = y(x), \quad x(T) = \int_0^T g(s)x(s)ds$$

in which ${}^cD^{\propto}_{0+}$ is a Caputo differential, $f:[0,T]\times R \to R, y: C^2([0,T],R) \to R$ and $g:[0,T]\to R$ are C^2 continuous function and $m \in R$.

Inspired by the aforementioned articles, we examine the solutions of the FDE below involving Caputo derivatives; ;

$${}^cD^{\gamma}_{0+}u(t) = g(t, u(t), {}^cD^{\delta}_{0+}u(t)) = 0, \ \ t \in [0,\tau], \ \ \gamma \in (n-1, n] \ \ where\ (n \ge 2)$$

subject to the nonlocal and integral boundary conditions;

$$k_0 = \mu(u), \ \ \int_0^{\tau} u(t)dt = \rho$$

$$k_0 = \mu(u), \quad u(\tau) = U_{\tau} \qquad (1.1)$$

Where ${}^cD^{\gamma}_{0+}$ and ${}^cD^{\delta}_{0+}$ are the Caputo fractional order differentials of γ and δ respectively, $g:[0,\tau]\times R\times R \to R$ is a continuous function, $\delta \in (0,1)$ for all $\delta, \rho, U_{\tau} \in R$.

As follows, we arrange the rest of the document. We stipulate our definitions and lemmas in section 2 to aid establish our main objectives. In section 3, by using the Banach and Schaefer's theorems of fixed points, we define some requirements for the existence of problem (1.1). Two examples to explain our key findings are discussed at the end.

II. PRELIMINARIES

To commence the section, we stipulate some definitions and notations utilized in our study and proceed to prove our supporting lemmas before stating our key findings.

Definition 2.1([11]) The Caputo fractional differential of order α for a continuous function $f(t)$ is defined by;

$${}^cD^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where n = $[\alpha]+1$, $[\alpha]$ connotes the integer of $\propto$ and Γ is the gamma function.

Definition 2.2([18]). If $f \in (0,\infty)$ is a continuous function, the Riemann-Liuoville fractional integral of order $\alpha > 0$ is defined as;

$$I^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds,$$

where Γ is the gamma function.

Definition 2.3([19]). The gamma function, $\Gamma(\cdot)$ is defined by,

$$\Gamma(\propto) = \int_0^{\infty} e^{-t}t^{\propto-1}dt,$$

where $[\alpha]$ is the integer of $\propto$ and $\Gamma(\propto+1) = \propto \Gamma(\propto)$.

Lemma 2.1([11]). If $\propto > 0$, the FDE $D^{\propto}_{0+}h(t) = 0$ has the solution $h(t) = k_0 + k_1t + k_2t^2 + \dots + k_{n-1}t^{n-1}, \ k_i \in R$, $i = 0,1,\dots,n-1$.

Lemma 2.2 ([19]). If $\propto > 0$, then $I^{\propto}_{0+}D^{\propto}_{0+}h(t) = h(t) + k_0 + k_1t + k_2t^2 + \dots + k_{n-1}t^{n-1}$ for any $k_i \in R$, $i = 0,1,\dots,n-1, \ n = [\propto]+1$.

Lemma 2.3. Let $\gamma \in (1,2]$ and $q(t) \in c^2[0,\tau]$. Then a function u satisfies the BVP;

$$\begin{cases} {}^cD^{\gamma}_{0+}u(t) = q(s), & t \in [0,\tau] \\ \mu(u) = k_0, & \int_0^{\tau} u(t)dt = \rho \end{cases} \qquad (2.1)$$

Iff u satisfies the fractional integral equation;

$$u(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}q(s)ds + (1-\frac{2t}{\tau})\mu(u) + \frac{2\rho}{\tau^2}t - \frac{2t}{\tau^2\Gamma(\gamma+1)}\int_0^\tau (\tau-s)^{\gamma}q(s)ds. \quad (2.2)$$

Proof: From lemma 2.2 with the boundary conditions $\mu(u) = k_0$ and $\int_0^\tau u(t)dt = \rho$, then;

$u(t) = I_{0+}^{\gamma}q(t) + k_0 + k_1 t$ where $k_0, k_1 \in R$,

$$u(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}q(s)ds + k_0 + k_1 t$$

$$\int_0^\tau u(t)dt = \frac{1}{\Gamma(\gamma)}\int_0^\tau\int_0^t (t-s)^{\gamma-1}q(s)ds + \int_0^\tau k_0 dt + \int_0^\tau k_1 t\, dt$$

$$2\rho = \frac{2}{\Gamma(\gamma+1)}\int_0^\tau (\tau-s)^{\gamma}q(s)ds + 2k_0\tau + \tau^2 k_1;$$

$$k_1 = \frac{2\rho}{\tau^2} - \frac{2}{\tau}\mu(u) - \frac{2}{\tau^2\Gamma(\gamma+1)}\int_0^\tau (\tau-s)^{\gamma}q(s)ds$$

Hence;

$$u(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}q(s)ds + (1-\frac{2t}{\tau})\mu(u) + \frac{2\rho}{\tau^2}t - \frac{2t}{\tau^2\Gamma(\gamma+1)}\int_0^\tau (\tau-s)^{\gamma}q(s)ds.$$

Prove is completed.

Lemma 2.4 Let $\gamma \in (1, 2]$ and $q: [0, \tau] \to R$ be continuous. Then u satisfies the fractional integral equation;

$$u(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}q(s)ds + (1-\frac{t}{\tau})\mu(u) + \frac{t}{\tau}U_\tau - \frac{t}{\tau\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1}q(s)ds \quad (2.3)$$

Iff u satisfies BVP (2.4);

$$\begin{cases} {}^{c}D_{0+}^{\gamma}u(t) = q(s), \quad t \in [0,\tau] \\ \mu(u) = k_0, \ u(\tau) = U_\tau \end{cases} \quad (2.4)$$

Proof: From lemma 2.2 and the boundary conditions $\mu(u) = k_0$, $u(\tau) = U_\tau$ then;

$u(t) = I_{0+}^{\gamma}q(t) + k_0 + k_1 t$ for some $k_0, k_1 \in R$,

$$u(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}q(s)ds + k_0 + k_1 t$$

this implies that;

$$u(\tau) = \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1}q(s)ds + k_0 + k_1\tau$$

$$U_\tau = \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1}q(s)ds + \mu(u) + k_1\tau$$

$$k_1 = \frac{U_\tau}{\tau} - \frac{1}{\tau}\mu(u) - \frac{1}{\tau\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1}q(s)ds$$

Hence,

$$u(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}q(s)ds - \frac{t}{\tau\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1}q(s)ds + (1-\frac{t}{\tau})\mu(u) + \frac{t}{\tau}U_\tau$$

Prove is completed.

III. MAIN RESULTS

Now we find appropriate to establish our main findings.

Theorem 3.1 Suppose;

$(A1)$. A non-negative constant L exists and satisfies;

$|g(t,u_1,v_1) - g(t,u_2,v_2)| \le L(|u_1-u_2| + |v_1-v_2|)$ *for any* $t \in [0,\tau]$ $\forall\, u_i, v_i \in R$.

(A_2) A non-negative constant L_1 exists and satisfies;

$|\mu(t,u) - \mu(t,v)| \le L_1|u-v|, \forall\, t \in [0,\tau]$ *and* $u, v \in R$.

(A_3) If $\varphi = \max\{(\frac{\tau^\gamma}{\Gamma(\gamma+1)} + \frac{2\tau^{\gamma-1}}{\Gamma(\gamma+2)})2L + L_1, \frac{\tau}{\Gamma(2-\delta)}[(\frac{\tau^{\gamma-1}}{\Gamma(\gamma)} + \frac{2\tau^{\gamma-2}}{\Gamma(\gamma+2)})2L + L_1]\} < 1.$

Then the BVP (2.1) has a unique solution.

Proof: We convert BVP (2.1) to a problem with fixed point by considering the operator $P : U \to U$, where $U \in C([0, \tau], R)$ defined by;

$$P(u)(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} g(s, u(s), {}^cD_{0+}^\delta u(s))ds - \frac{2t}{\tau^2\Gamma(\gamma+1)}\int_0^\tau (\tau-s)^\gamma g(s, u(s), {}^cD_{0+}^\delta u(s))ds$$
$$+(1-\frac{2t}{\tau})\mu(u) + \frac{2\rho}{\tau^2}t \qquad (2.5)$$

Obviously, P has fixed points which satisfies problem (2.1). Now, from (A_1) we show that P has a fixed point and hence maps onto itself.

Let $u_1, u_2 \in u$, then;

$$|P(u_1)(t) - P(u_2)(t)| \le \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}|g(s, u_1(s), {}^cD_{0+}^\delta u_1(s)) - g(s, u_1(s), {}^cD_{0+}^\delta u_2(s))|ds$$
$$+\frac{2t}{\tau^2\Gamma(\gamma+1)}\int_0^\tau (\tau-s)^\gamma |g(s, u_1(s), {}^cD_{0+}^\delta u_1(s)) - g(s, u_1(s), {}^cD_{0+}^\delta u_2(s))ds|$$
$$+\left|1-\frac{2t}{\tau}\right| |\mu(u_1) - \mu(u_2)|. \qquad (2.6)$$

From (A_1), it implies that;

$$\left|g(s, u_1(s), {}^cD_{0+}^\delta u_1(s)) - g(s, u_1(s), {}^cD_{0+}^\delta u_2(s))\right| \le L(|u_1(s) - u_2(s)| + \left|{}^cD_{0+}^\delta u_1(s)) - {}^cD_{0+}^\delta u_2(s))\right|$$
$$\le L(\|u_1 - u_2\| + \left\|{}^cD_{0+}^\delta u_1 - {}^cD_{0+}^\delta u_2\right\|$$
$$\le 2L\|u_1 - u_2\|_* \qquad (2.7)$$

And from (A_2),

$$|\mu(u_1) - \mu(u_2)| \le L_1\|u_1 - u_2\|_* \qquad (2.8)$$

Therefore from (2.6),

$$|P(u_1)(t) - P(u_2)(t)| \le \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}ds.\, 2L\|u_1 - u_2\|_* + \left|1-\frac{2t}{\tau}\right|.\, L_1\|u_1 - u_2\|_*$$
$$+\frac{2t}{\tau^2\Gamma(\gamma+1)}\int_0^\tau (\tau-s)^\gamma ds.\, 2L\|u_1 - u_2\|_*$$

$$\|P(u_1) - P(u_2)\| \le \frac{\tau^\gamma}{\gamma\Gamma(\gamma)}.\, 2L\|u_1 - u_2\|_* + L_1\|u_1 - u_2\|_* + \frac{2}{\tau(\gamma+1)\Gamma(\gamma+1)}\tau^\gamma.\, 2L\|u_1 - u_2\|_*$$
$$\le (\frac{2L\tau^\gamma}{\Gamma(\gamma+1)} + \frac{4L\tau^{\gamma-1}}{\Gamma(\gamma+2)}).\, \|u_1 - u_2\|_* + L_1\|u_1 - u_2\|_*$$
$$\le [(\frac{\tau^\gamma}{\Gamma(\gamma+1)} + \frac{2\tau^{\gamma-1}}{\Gamma(\gamma+2)})2L + L_1]\|u_1 - u_2\|_*$$

Again from (2.5);

$$(Pu)'(t) = \frac{1}{\Gamma(\gamma-1)}\int_0^t (t-s)^{\gamma-2} g(s, u(s), {}^cD_{0+}^\delta u(s))ds - \frac{2}{\tau}\mu(u) + \frac{2\rho}{\tau^2} - \frac{2}{\tau^2\Gamma(\gamma+1)}\int_0^\tau (\tau-s)^\gamma g(s, u(s), {}^cD_{0+}^\delta u(s)ds$$

Then;

$$\left|(Pu_1)'(t) - (Pu_2)'(t)\right| \le \frac{1}{\Gamma(\gamma-1)}\int_0^t (t-s)^{\gamma-2}\left|g(s, u_1(s), {}^cD_{0+}^\delta u_1(s)) - g(s, u_2(s), {}^cD_{0+}^\delta u_2(s))\right|ds$$
$$+\frac{2}{\tau^2\Gamma(\gamma+1)}\int_0^\tau (\tau-s)^\gamma \left|g(s, u_1(s), {}^cD_{0+}^\delta u_1(s)) - g(s, u_2(s), {}^cD_{0+}^\delta u_2(s))\right|ds$$
$$+|\mu(u_1) - \mu(u_2)|$$

From (2.7) and (2.8),

$$\left|(Pu_1)'(t) - (Pu_2)'(t)\right| \le \frac{t^{\gamma-1}}{\Gamma(\gamma)}.\, 2L\|u_1 - u_2\|_* + L_1\|u_1 - u_2\|_* + \frac{2\tau^{\gamma-2}}{\Gamma(\gamma+2)}.\, 2L\|u_1 - u_2\|_*$$

$$\leq [(\frac{\tau^{\gamma-1}}{\Gamma(\gamma)} + \frac{2\tau^{\gamma-2}}{\Gamma(\gamma+2)})2L + L_1] . \|u_1 - u_2\|_*$$

But;

$${}^{c}D^{\delta}_{0+}(Pu)'(t) = \frac{1}{\Gamma(1-\delta)} \int_0^t (t-s)^{-\delta} ds. (Pu)'(t)$$

$$= \frac{1}{\Gamma(1-\delta)} . \frac{t^{1-\delta}}{1-\delta} (Pu)'(t)$$

$$= \frac{1}{\Gamma(2-\delta)} . (Pu)'(t)$$

Therefore;

$$\left|{}^{c}D^{\delta}_{0+}(Pu_1)'(t) - {}^{c}D^{\delta}_{0+}(Pu_2)'(t)\right| \leq \frac{1}{\Gamma(1-\delta)} \int_0^t (t-s)^{-\delta} ds \left|(Pu_1)'(s) - (Pu_2)'(s)\right|$$

$$\leq \frac{\tau}{\Gamma(2-\delta)} [(\frac{\tau^{\gamma-1}}{\Gamma(\gamma)} + \frac{2\tau^{\gamma-2}}{\Gamma(\gamma+2)})2L + L_1] \|u_1 - u_2\|_*$$

Hence,

$$\|Pu_1 - Pu_2\|_* = max\{\|P(u_1) - P(u_2)\|, \left\|{}^{c}D^{\delta}_{0+}(Pu_1)'(t) - {}^{c}D^{\delta}_{0+}(Pu_2)'(t)\right\|\} \leq \varphi \|u_1 - u_2\|_*$$

Hence, P with $\varphi \leq 1$ contracts. Therefore as a result of the contraction principle, we infer that P has a fixed point which is unique and satisfies (2.1). Consequently, the proof is completed.

Theorem 3.2 Suppose $g : [0, \tau] \times R \times R \to R$ and $\mu : C([0,1], R) \to R$ are continuous functions and satisfy;

(B_1) $|g(t, u, v)| \leq k$ for any $t \in [0, \tau]$ $\forall$ u, v $\in$ R,

(B_2) $|\mu(t, u) - \mu(t, v)| \leq k_1|u - v|$ for any $t \in [0, \tau]$ $\forall$ u, v $\in$ R.

(B_3) $|\mu(u)| \leq k_*$ $\forall$ $\mu \in C([0, \tau], R)$.

where k and k_* are positive constants, then BVP (2.1) has a solution.

Proof: To ascertain our results, we use the Schaefer's fixed point theorem. We divide the proof into four parts for simplicity and clarity.

Step 1: Firstly, we establish the continuity of P.

Let $u_n \in C([0, \tau], R)$ be a sequence such that $u_n \to u$. Then $\forall$ $t \in [0, \tau]$, we derive;

$$P(u_n)(t) - P(u)(t) \leq \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \left|g(s, u_n(s), {}^{c}D^{\delta}_{0+}u_n(s)) - g(s, u(s), {}^{c}D^{\delta}_{0+}u(s))\right| ds$$

$$+ \frac{2}{\tau^2 \Gamma(\gamma+1)} \int_0^\tau (\tau-s)^{\gamma} \left|g(s, u_n(s), {}^{c}D^{\delta}_{0+}u_n(s)) - g(s, u(s), {}^{c}D^{\delta}_{0+}u(s))\right| ds$$

$$+ |\mu(u_n) - \mu(u)|$$

From (B_1) and (B_2);

$$\left|g(t, u_n(t), {}^{c}D^{\delta}_{0+}u_n(t)) - g(t, u(t), {}^{c}D^{\delta}_{0+}u(t))\right| \leq k(\|u_n(s) - u(s)\| + \left\|{}^{c}D^{\delta}_{0+}u_n(s) - {}^{c}D^{\delta}_{0+}u(s)\right\|)$$

$$\leq k(\|u_n - u\| + \left\|{}^{c}D^{\delta}_{0+}u_n - {}^{c}D^{\delta}_{0+}u\right\|)$$

$$\leq 2k\|u_n - u\|_*$$

and

$$\|\mu(u_n) - \mu(u)\| \leq k_1\|u_n - u\|$$

Therefore,

$$|P(u_n)(t) - P(u)(t)| \leq \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} ds.\ 2k\|u_n - u\|_* + k_1\|u_n - u\| + \frac{2}{T^2 \Gamma(\gamma+1)} \int_0^T (T-s)^{\gamma} ds. 2k\|u_n - u\|_*$$

also,

$$\left|{}^{c}D^{\delta}_{0+}P(u_n)(t) - {}^{c}D^{\delta}_{0+}P(u)(t)\right| \leq \frac{1}{\Gamma(1-\delta)} \int_0^t (t-s)^{-\delta} |P(u_n)(s) - (Pu)(s)| ds$$

Since the functions g and μ are continuous, it implies that $\|P(u_n) - P(u)\| \to 0$ and $\|{}^cD_{0+}^{\delta}P(u_n) - {}^cD_{0+}^{\delta}P(u)\| \to 0$ as $n \to \infty$. Therefore P is continuous.

Step 2: Now we prove the boundedness of P.

By (B_1) *and* (B_3) from (2.5) for any $t \in [0, \tau]$, we derive;

$$|P(u)(t)| \leq \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\left|g(s,u(s),{}^cD_{0+}^{\delta}u(s))\right|ds + 2|\mu(u)| + \frac{2}{\tau}|\rho| + \frac{2}{\tau\Gamma(\gamma+1)}\int_0^{\tau}(\tau-s)^{\gamma}\left|g(s,u(s),{}^cD_{0+}^{\delta}u(s))\right|ds$$

$$\leq \frac{k}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}ds + 2k_* + \frac{2}{\tau}|\rho| + \frac{2k}{\tau\Gamma(\gamma+1)}\int_0^{\tau}(\tau-s)^{\gamma}ds$$

$$\leq \frac{k}{\gamma\Gamma(\gamma)}.\tau^{\gamma} + \frac{2k}{\tau}.\frac{\tau^{\gamma+1}}{(\gamma+1)\Gamma(\gamma+1)} + 2k_* + \frac{2}{\tau}|\rho|$$

$$\leq \frac{k}{\Gamma(\gamma+1)}.\tau^{\gamma} + \frac{2k}{\Gamma(\gamma+2)}.\tau^{\gamma} + 2k_* + \frac{2}{\tau}|\rho|$$

and,

$$\left|P({}^cD_{0+}^{\delta}u(t))\right| = \left|{}^cD_{0+}^{\delta}P(u)(s)\right| \leq \frac{1}{\Gamma(1-\delta)}\int_0^t (t-s)^{-\delta}ds.\,|(Pu)(s)|$$

$$\leq \frac{\tau}{\Gamma(2-\delta)}\left[\frac{k}{\Gamma(\gamma+1)}.\tau^{\gamma} + \frac{2k}{\Gamma(\gamma+2)}.\tau^{\gamma} + 2k_* + \frac{2}{\tau}|\rho|\right.$$

Thus, P is evenly bounded.

Step 3: Next we establish that P is completely continuous.

Let $t_1, t_2 \in (0, \tau]$ where $t_1 < t_2$, then;

$$|P(u)(t_2) - P(u)(t_1)| =$$

$$\left|\frac{1}{\Gamma(\gamma)}\int_0^{t_1}[(t_2-s)^{\gamma-1} - (t_1-s)^{\gamma-1}]g(s,u(s),{}^cD_{0+}^{\delta}u(s))\,ds + \frac{1}{\Gamma(\gamma)}\int_{t_1}^{t_2}[(t_2-s)^{\gamma-1}g(s,u(s),{}^cD_{0+}^{\delta}u(s))\right|ds$$

$$+\frac{2(t_2-t_1)}{\tau}|\mu(u)| + \frac{2(t_2-t_1)}{\tau^2}|\rho| + \frac{2(t_2-t_1)}{\tau^2\Gamma(\gamma+1)}\int_0^{\tau}(\tau-s)^{\gamma}\left|g(s,u(s),{}^cD_{0+}^{\delta}u(s))\right|ds$$

$$\leq \frac{k}{\Gamma(\gamma)}\int_0^{t_1}[(t_2-s)^{\gamma-1} - (t_1-s)^{\gamma-1}]ds + \frac{k}{\Gamma(\gamma)}\int_{t_1}^{t_2}(t_2-s)^{\gamma-1}ds + \frac{2k_*}{T}(t_2-t_1) + \frac{2|\rho|}{T^2}(t_2-t_1)$$

$$+\frac{2k}{T^2\Gamma(\gamma+1)}\int_0^{\tau}(\tau-s)^{\gamma}ds.\,(t_2-t_1)$$

$$\leq \frac{k}{\Gamma(\gamma+1)}[(t_2-t_1)^{\gamma} + t_2^{\gamma} - t_1^{\gamma}] + \frac{k}{\Gamma(\gamma+1)}(t_2-t_1)^{\gamma} + \frac{2k\tau^{\gamma+1}}{\tau^2\Gamma(\gamma+2)}(t_2-t_1) + \frac{2k_*}{\tau}(t_2-t_1) + \frac{2|\rho|}{\tau^2}(t_2-t_1) \quad (2.9)$$

From (2.9), it can be easily seen that as $t_1 \to t_2$, the RHS tends to zero. Together with the Arzela - Ascoli theorem and the preceding steps, we conclude that P is completely continuous.

Step 4: Finally, we assume P maps onto itself, such that the set $\Phi = \{u \in E: u = \lambda Pu \text{ for any } \lambda \in (0,1)\}$ is bounded.

Now $\forall\, t \in [0, \tau]$,

$$P(u)(t) = \frac{\lambda}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}g(s,u(s),{}^cD_{0+}^{\delta}u(s))ds + \lambda(1-\frac{t}{\tau})\mu(u) + \lambda\frac{2t}{\tau^2}\rho$$

$$-\lambda\frac{2t}{\tau^2\Gamma(\gamma+1)}\int_0^{\tau}(\tau-s)^{\gamma}.\,g(s,u(s),{}^cD_{0+}^{\delta}u(s))ds$$

By (B_1) and (B_3) for any $t \in [0,\tau]$;

$$|P(u)(t)| \leq \frac{k}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}ds + k_* + \frac{2}{\tau}|\rho| + \frac{2k}{\tau\Gamma(\gamma+1)}\int_0^{\tau}(\tau-s)^{\gamma}ds$$

$$\leq \frac{k}{\Gamma(\gamma+1)}\tau^{\gamma} + \frac{2k}{\Gamma(\gamma+2)}\tau^{\gamma+1} + k_* + \frac{2}{\tau}|\rho|$$

Hence the set Φ is bounded. Therefore P has a fixed point which is a solution to problem(2.1). The end of prove.

At this point, we examine the solutions of BVP (2.4) to ascertain its existence and uniqueness.

Theorem 3.3 Suppose;

(C_1) A non-negative constant L exists and satisfies;

$|g(t,u_1,v_1)-g(t,u_2,v_2)| \leq L(|u_1-u_2|+|v_1-v_2|)$ for each $t \in J$ and all $u_i, v_i \in R$.

(C_2) A constant $L_1 > 0$ exists and satisfies condition;

$|\mu(u_0)-\mu(u_1)| < L_1|u_0-u_1|$, for any $t \in J$ and all $u_0, u_1 \in U$

(C_3) If $\varphi = \max\{(\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}+\frac{\tau^{\gamma-1}}{\Gamma(\gamma+1)})2L+L_1, \frac{\tau}{\Gamma(2-\beta)}[(\frac{\tau^{\gamma-1}}{\Gamma(\gamma)}+\frac{\tau^{\gamma-1}}{\Gamma(\gamma+1)})2L+L_1]\} < 1$.

Then, the BVP (2.4) has a unique solution.

Proof: We convert BVP (2.4) to a problem with fixed point by considering the operator $P: U \to U$ defined by;

$$P(u)(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} g(s,u(s),{}^cD_{0+}^{\delta}u(s))ds + (1-\frac{t}{\tau})\mu(u)+\frac{t}{\tau}U_\tau$$
$$-\frac{t}{\tau\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1} g(s,u(s),{}^cD_{0+}^{\delta}u(s))ds \tag{2.10}$$

Now we show that P has a fixed point and hence is a contraction.

Let $u_1, u_2 \in U \; \forall \; t \in [0,\tau]$;

$$|P(u_1)(t)-P(u_2)(t)| \leq \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\left|g(s,u_1(s),{}^cD_{0+}^{\delta}u_1(s))-g(s,u_2(s),{}^cD_{0+}^{\delta}u_2(s))\right|ds$$
$$+\frac{t}{\tau\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1}\left|g(s,u_1(s),{}^cD_{0+}^{\delta}u_1(s))-g(s,u_2(s),{}^cD_{0+}^{\delta}u_2(s))\right|ds$$
$$+|\mu(u_1)-\mu(u_2)|$$

From (C_1) and (C_2);

$$\left|g(t,u_1(t),{}^cD_{0+}^{\delta}u_1(t))-g(t,u_2(t),{}^cD_{0+}^{\delta}u_2(t))\right| \leq L(|u_1(s)-u_2(s)|+\left|{}^cD_{0+}^{\delta}u_1(s)-{}^cD_{0+}^{\delta}u_2(s)\right|)$$
$$\leq L(\|u_1-u_2\|+\left\|{}^cD_{0+}^{\delta}u_1-{}^cD_{0+}^{\delta}u_2\right\|)$$
$$\leq 2L\|u_1-u_2\|_1 \tag{2.11}$$

And,

$$|\mu(u_1)(t)-\mu(u_2)(t)| \leq L_1\|u_1(s)-u_2(s)\|$$
$$\leq L_1\|u_1-u_2\|_1 \tag{2.12}$$

From (2.11) and (2.12),

$$|P(u_1)(t)-P(u_2)(t)| \leq \frac{\tau^{\gamma}}{\Gamma(\gamma+1)}.2L\|u_1-u_2\|_1+\frac{\tau^{\gamma-1}}{\Gamma(\gamma+1)}.2L\|u_1-u_2\|_1+L_1\|u_1-u_2\|_1$$
$$\leq [(\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}+\frac{\tau^{\gamma-1}}{\Gamma(\gamma+1)})2L+L_1]\|u_1-u_2\|_1$$

Again from (2.10),

$$(Pu)'(t) = \frac{1}{\Gamma(\gamma-1)}\int_0^t (t-s)^{\gamma-2} g(s,u(s),{}^cD_{0+}^{\delta}u(s))ds - \frac{1}{\tau}\mu(u)+\frac{1}{\tau}U_\tau - \frac{1}{\tau\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1} g(s,u(s),{}^cD_{0+}^{\delta}u(s))ds$$

Then;

$$\left|(Pu_1)'(t)-(Pu_2)'(t)\right| \leq \frac{1}{\Gamma(\gamma-1)}\int_0^t (t-s)^{\gamma-2}\left|g(s,u_1(s),{}^cD_{0+}^{\delta}u_1(s))-g(s,u_2(s),{}^cD_{0+}^{\delta}u_2(s))\right|ds$$
$$+\frac{1}{\tau\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1}\left|g(s,u_1(s),{}^cD_{0+}^{\delta}u_1(s))-g(s,u_2(s),{}^cD_{0+}^{\delta}u_2(s))\right|ds$$
$$+\frac{1}{\tau}|\mu(u_1)-\mu(u_2)|$$

From (2.11) and (2.12),

$$\|(Pu_1)' - (Pu_2)'\| \leq \frac{\tau^{\gamma-1}}{\Gamma(\gamma)} 2L\|u_1 - u_2\|_1 + L_1\|u_1 - u_2\|_1 + \frac{\tau^{\gamma}}{\Gamma(\gamma+1)} 2L\|u_1 - u_2\|_1$$

$$\leq [(\frac{\tau^{\gamma-1}}{\Gamma(\gamma)} + \frac{\tau^{\gamma}}{\Gamma(\gamma+1)})2L + L_1]\|u_1 - u_2\|_1$$

and

$$\left|{}^cD^{\delta}_{0+}(Pu_1)(t) - {}^cD^{\delta}_{0+}(Pu_2)(t)\right| \leq \frac{1}{\Gamma(1-\delta)}\int_0^t (t-s)^{-\delta}\left|(Pu_1)'(t) - (Pu_2)'(t)\right| ds$$

$$\leq \frac{\tau}{(2-\delta)}[(\frac{\tau^{\gamma-1}}{\Gamma(\gamma)} + \frac{\tau^{\gamma}}{\Gamma(\gamma+1)})2L + L_1]\|u_1 - u_2\|_1$$

Therefore;

$$\|P(u_1) - P(u_2)\|_1 = max\{\|P(u_1) - P(u_2)\|, \|{}^cD^{\delta}P(u_1) - {}^cD^{\delta}P(u_2)\|\} \leq \varphi\|u_1 - u_2\|_1$$

Therefore with $\varphi < 1$, P is said to contract and hence uniquely satisfies the BVP (2.4). Thus, the end of proof..

Theorem 3.4 Suppose $g : [0, \tau] \times R \times R \to R$ and $\mu : C([0, 1], R) \to R$ are continuous functions and satisfy;

(D_1) $|g(t, u, v)| \leq k$ for any $t \in J$ $\forall$ $u, v \in R$.

(D_2) $|\mu(u)| \leq k_1$ $\forall$ $u \in C([0, \tau], R)$.

where k and k_1 are positive constants, then BVP (2.4) has a solution.

Proof: Similarly, we apply the Schaefer's theorem of fixed points in proving our results.

Step 1: We prove that P is continuous. Let the sequence u_n exist in such a way that $u_n \to u \in U$. For any $t \in [0, \tau]$;

$$|P(u_n)(t) - P(u)(t)| \leq \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\left|g(s, u_n(s), {}^cD^{\delta}u_n(s)) - g(s, u(s), {}^cD^{\delta}u(s))\right| ds + |\mu(u_n) - \mu(u)|$$

$$+ \frac{1}{\tau\Gamma(\gamma)}\int_0^{\tau} (\tau-s)^{\gamma-1}\left|g(s, u_n(s), {}^cD^{\delta}u_n(s)) - g(s, u(s), {}^cD^{\delta}u(s))\right| ds$$

From assumptions (D_2) and (D_3);

$$\left|g(s, u_n(s), {}^cD^{\delta}u_n(s)) - g(s, u(s), {}^cD^{\delta}u(s))\right| \leq 2k\|u_n - u\|_2$$

and

$$|\mu(u_n) - \mu(u)| \leq k_1\|u_n - u\|_2$$

Therefore;

$$|P(u_n)(t) - P(u)(t)| \leq \frac{\tau^{\gamma}}{\Gamma(\gamma+1)} 2k\|u_n - u\|_2 + k_1\|u_n - u\|_2 + \frac{\tau^{\gamma-1}}{\Gamma(\gamma+1)} 2k\|u_n - u\|_2$$

also,

$$\left|{}^cD^{\delta}P(u_n)(t) - {}^cD^{\delta}P(u)(t)\right| \leq \frac{1}{\Gamma(1-\delta)}\int_0^t (t-s)^{-\delta}\,|P(u_n)(s) - P(u)(s)| ds$$

From the above, we conclude that P is continuous since the functions g and μ are continuous which implies that $\|P(u_n) - P(u)\| \to 0$ and $\|{}^cD^{\delta}P(u_n)(t) - {}^cD^{\delta}P(u)(t)\| \to 0$ as n approaches infinity.

Step 2: P transforms bounded sets into another sets bounded in U. A non-negative constant r exists and satisfies $\|P(u)\| \leq r$. Hence, for $t \in [0, \tau]$ we derive;

By (D_1) and (D_2);

$$|P(u)(t)| \leq \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\left|g(s, u(s), {}^cD^{\delta}u(s))\right| ds + |\mu(u)| + |U_{\tau}| + \frac{1}{\Gamma(\gamma)}\int_0^{\tau} (\tau-s)^{\gamma-1}\left|g(s, u(s), {}^cD^{\delta}u(s))\right| ds$$

$$\leq \frac{k}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} ds + k_1 + |U_{\tau}| + \frac{k}{\Gamma(\gamma)}\int_0^{\tau} (\tau-s)^{\gamma-1} ds$$

$$\leq \frac{k}{\Gamma(\gamma+1)}\tau^{\gamma} + \frac{k}{\Gamma(\gamma+1)}\tau^{\gamma} + k_1 + |U_{\tau}|$$

Therefore;

$\|P(u)\| \leq \frac{2\tau^{\gamma}}{\Gamma(\gamma+1)}k + k_1 + |U_\tau| = r$

Step 3: P maps any bounded sets into an equi-continuous sets in U.

Let $t_1, t_2 \in [0,\tau]$, $t_1 \leq t_2$. Then from (2.10) we have;

$$|P(u)(t_2) - P(u)(t_1)| = \frac{1}{\Gamma(\gamma)}\left|\int_0^{t_1}[(t_2-s)^{\gamma-1} - (t_1-s)^{\gamma-1}]\, g(s,u(s),{}^cD^{\delta}u(s))\right| ds$$
$$+\frac{1}{\Gamma(\gamma)}\left|\int_{t_1}^{t_2}(t-s)^{\gamma-1}g(s,u(s),{}^cD^{\delta}u(s))\right| ds + \frac{(t_2-t_1)}{\tau}|\mu(u)|$$
$$+\frac{(t_2-t_1)}{\tau}|U_\tau| + \frac{(t_2-t_1)}{\tau\Gamma(\gamma)}\int_0^{\tau}(\tau-s)^{\gamma-1}|g(s,u(s),{}^cD^{\delta}u(s))|ds$$
$$\leq \frac{k}{\Gamma(\gamma)}\int_0^{t_1}[(t_2-s)^{\gamma-1} - (t_1-s)^{\gamma-1}]\,ds + \frac{k}{\Gamma(\gamma)}\int_{t_1}^{t_2}(t-s)^{\gamma-1}ds + \frac{(t_2-t_1)}{\tau}k_1$$
$$+\frac{(t_2-t_1)}{\tau}|U_\tau| + \frac{k(t_2-t_1)}{\tau\Gamma(\gamma)}\int_0^{\tau}(\tau-s)^{\gamma-1}ds$$
$$\leq \frac{M}{\Gamma(\gamma+1)}[(t_2-t_1)^{\gamma} + t_2^{\gamma} - t_1^{\gamma}] + \frac{k}{\Gamma(\gamma+1)}(t_2-t_1)^{\gamma} + \frac{(t_2-t_1)}{\tau}k_1 + \frac{(t_2-t_1)}{\tau}|U_\tau| + \frac{k\tau^{\gamma-1}}{\Gamma(\gamma+1)}(t_2-t_1) \quad (2.13)$$

As $t_1 \to t_2$, the RHS of (2.13) tends to zero. Therefore P is completely continuous as a consequence of the above steps and the Arzela-Ascoli theorem.

Step 4: Finally, we prove the set $\phi = \{u \in E;\ u = \lambda Pu \text{ for any } \lambda \in (0,1)\}$

Let $u \in \phi$, then

$u = \lambda P(u)\ \forall\ \lambda \in (0,1)$ and $0 \leq t \leq \tau$,

$$P(u)(t) = \frac{\lambda}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}g(s,u(s),{}^cD^{\delta}u(s)) + \lambda(1-\frac{t}{\tau})\mu(u) - \frac{\lambda t}{\tau}U_\tau - \frac{\lambda t}{\tau\Gamma(\gamma)}\int_0^{\tau}(\tau-s)^{\gamma-1}g(s,u(s),{}^cD^{\delta}u(s))ds$$

From (D_2) and (D_3);

$$|P(u)(t)| \leq \frac{k}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}ds + k_1 + |U_\tau| + \frac{k}{\Gamma(\gamma)}\int_0^{\tau}(\tau-s)^{\gamma-1}ds$$
$$\leq \frac{k}{\Gamma(\gamma+1)}.\tau^{\gamma} + \frac{k}{\Gamma(\gamma+1)}.\tau^{\gamma} + k_1 + |U_\tau|$$
$$\|P(u)\| \leq \frac{\tau^{\gamma}}{\Gamma(\gamma+1)}.2k + k_1 + |U_\tau|$$

This shows that the set ϕ is bounded and hence satisfies problem (2.10).

IV. APPLICATION

At this point, we apply our obtained results to some selected examples.

Example 4.1 Consider the fractional boundary value problem;

$${}^cD_{0+}^{1.5}u(t) = t + 0.1tu(t) + (0.1t^2)\,{}^cD_{0+}^{0.5}u(t); \quad t \in (0,1], \quad \gamma \in (1,2],$$
$$k_0 = \sum_{i=1}^{n} c_i u(t_i), \quad \int_0^{\tau} u(t)dt = 1$$

Where $t_i \in (0,1)$, c_i, $i = 1,2,\ldots,n-1$, n are non-negative constants with $\sum_{i=1}^{n} c_i < \frac{1}{2}$. Set $\gamma = 1.5$ $(n \geq 2)$, $\delta = 0.5$, $\mu(u) = \sum_{i=1}^{n} c_i u(t_i)$; $\tau = 1$, $\rho = 1$ and

$g(t,u(t),{}^cD_{0+}^{\delta}u(t)) = t + 0.1tu(t) + (0.1t^2)\,{}^cD_{0+}^{0.5}u(t)$.

Let $t \in [0,\tau]$ and $u_i, v_i \in R$, where $i = 1, 2$.

Then we have;

$$|g(t,u_1,u_2) - g(t,v_1,v_2)| = |0.1t(u_1) - 0.1t(u_2)| + |0.1t^2(v_1) - 0.1t^2(v_2)|$$
$$\leq 0.1(|u_1 - u_2| + |v_1 - v_2|)$$

Hence, satisfies condition (A_1) with $L = 0.1$. Also,

$$|\mu(u_1) - \mu(u_2)| = \left|\sum_{i=1}^{n} c_i u_1(t_i) - \sum_{i=1}^{n} c_i u_2(t_i)\right|$$
$$\leq \sum_{i=1}^{n} c_i \|u_1 - u_2\|$$

Hence, (A_1) is satisfied with $L_1 = \sum_{i=1}^{n} c_i < \frac{1}{2}$

Finally, we show that (A_3) holds.

$$\varphi = \left(\frac{\tau^{\gamma}}{\Gamma(\gamma+1)} + \frac{2\tau^{\gamma-1}}{\Gamma(\gamma+2)}\right)2L + L_1 = \left(\frac{1}{\Gamma(2.5)} + \frac{2}{\Gamma(3.5)}\right)2(0.1) + 0.5 \cong 0.771 < 1.$$

and

$$\frac{\tau}{\Gamma(2-\delta)}\left[\left(\frac{\tau^{\gamma-1}}{\Gamma(\gamma)} + \frac{2\tau^{\gamma-2}}{\Gamma(\gamma+2)}\right)2L + L_1 = \frac{1}{\Gamma(1.5)}\left[\left(\frac{1}{\Gamma(1.5)} + \frac{2}{\Gamma(3.5)}\right)\right] 2(0.1) + 0.5 \cong 0.954.\right.$$

(A_3) is satisfied with $\varphi < 1$.

Hence, from theorem 3.1 we conclude that BVP (2.1) has a unique solution.

Example 4.2 Consider the fractional boundary value problem;

$$^{c}D_{0^+}^{1.5}u(t) = e^{\frac{-1}{2}t} + \frac{e^{\frac{-1}{2}t}u(t)}{10e^{t}(1+u(t))} + \left(\frac{e^{-t^3}}{10e^{t}}\right){}^{c}D_{0^+}^{0.5}u(t)$$

$$u(0) = \sum_{i=1}^{n} c_i u(t_i); \quad u(1) = 0$$

Where $t_i \in (0, 1)$, $c_{i,}$ $i = 1,2, \dots, n-1, n$ are non-negative constants with

$\sum_{i=1}^{n} c_i < \frac{1}{6}$. From BVP (3.2), $\gamma = 1.5$, $\delta = 0.5$, $\tau = 1$, $U_\tau = 0$,

$$g(t, u(t), {}^{c}D^{\delta}u(t)) = e^{\frac{-1}{2}t} + \frac{e^{\frac{-1}{2}t}u(t)}{10e^{t}(1+u(t))} + \left(\frac{e^{-t^3}}{10e^{t}}\right){}^{c}D_{0^+}^{0.5}u(t), \quad \mu(u) = \sum_{i=1}^{n} c_i u(t_i);$$

Let $u_i, v_i \in R, i = 1,2$ and $t \in [0, 1]$. Then;

$$|g(t, u_1, v_1) - g(t, u_2, v_2)| = \left|\frac{e^{-1/2t^2}}{10e^{t}}u_1(t) - \frac{e^{-1/2t^2}}{10e^{t}}u_2(t)\right| + \left|\frac{e^{-1/2t^2}}{10e^{t}}v_1(t) - \frac{e^{-1/2t^2}}{10e^{t}}v_2(t)\right|$$
$$= \frac{1}{10}(|u_1 - u_2| + |v_1 - v_2|)$$

Hence, satisfies condition (C_1) with $L = \frac{1}{10}$.

Also,

$$|\mu(u) - \mu(v)| \leq \sum_{i=1}^{n} c_i |u - v|$$

Now, we verify that (C_3) is satisfied with $\tau = 1$.

$$\left(\frac{\tau^{\gamma}}{\Gamma(\gamma+1)} + \frac{\tau^{\gamma-1}}{\Gamma(\gamma+1)}\right)2L + L_1 = \left(\frac{2}{\Gamma(2.5)}\right) \times 2(0.1) + \frac{1}{6} \cong 0.4675 < 1.$$

and,

$$\frac{\tau}{\Gamma(2-\beta)}\left[\left(\frac{\tau^{\gamma-1}}{\Gamma(\gamma)} + \frac{\tau^{\gamma}}{\Gamma(\gamma+1)}\right)2L + L_1\right] = \frac{1}{\Gamma(1.5)}\left[\left(\frac{1}{\Gamma(1.5)} + \frac{1}{\Gamma(2.5)}\right) \times 2(0.1) + \frac{1}{6}\right] \cong 0.6123 < 1$$

which satisfies (C_3) with $\varphi < 1$ for any $\gamma \in (1, 2]$. Hence, we establish that BVP (2.4) has a unique solution.

V. CONCLUSION

This article examines the existence and uniqueness of solutions using the principle of contraction mapping and the Schaefer's fixed point theorem to ascertain our results for fractional differential equations involving Caputo derivatives with nonlocal and integral boundary conditions. Boundary problems for fractional derivatives with multiple boundary conditions or eigenvalue problems may be considered for further research works.

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