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Analysis of Wave Constants in the Laplace Equation Solution for Deep Water with Wave Amplitude or Wave Energy as Input

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Abstract*— *In the velocity potential solution of the Laplace equation obtained using the separation of variables method, three wave constants arise: wavelength, wave period, and the wave constant G. The wave constant G represents the rate of wave energy transmission. Unlike these constants, wave amplitude is not part of the solution constants but serves as an input parameter. Therefore, the wave constants should be expressed as functions of the wave amplitude. This study derives analytical expressions for wavelength, wave constant G, and wave period in deep water with wave amplitude as the governing variable. The relationship between wavelength and wave amplitude is obtained using the Kinematic Free Surface Boundary Condition. The relationship between wave constant G and wave amplitude is derived from a modified Euler momentum conservation equation together with a wave amplitude function, which relates the three wave constants to the wave amplitude. The wave period is then determined using the equations for wave constant G and the wave amplitude function. After establishing the wave constants as functions of wave amplitude, the study further formulates wave amplitude as a function of wave energy. The resulting amplitude is then used to calculate the three wave constants. This approach can also be applied to analyze waves generated by ship motion, where the input energy corresponds to the ship's kinetic energy. The method is further extended to long waves, particularly tsunamis and sneaker waves.

I. INTRODUCTION

The velocity potential solution of the Laplace equation obtained through the separation of variables method, as presented by Dean (1991), contains three wave constants. To apply this solution in subsequent wave analysis, the values of these constants must first be determined.

Within linear wave theory, Dean (1991) derived expressions for the wave constants using wave amplitude and wave period as input parameters. In this formulation, wave amplitude and wave period are treated as independent variables, implying that no inherent relationship exists between them.

However, several earlier studies have indicated a relationship between wave height and wave period. For instance, Robert L. Wiegel (1949, 1964) developed an empirical relationship between wave height and wave period for deep-water waves. Later, Willard J. Pierson Jr. and Lionel Moskowitz (1962) introduced the Pierson–

Moskowitz spectrum, which describes the distribution of wave energy with respect to wave period. Based on this spectrum, Robert Silvester (1974) subsequently formulated an equation relating wave period to wave height.

More recently, Hutahaean (2024) applied the kinematic free surface boundary condition to derive a wave amplitude function equation that relates wave amplitude to the three wave constants. By combining this function with the Euler equations for momentum conservation while neglecting convective acceleration, a relationship between wave period and wave amplitude was obtained.

These previous studies suggest that wave period should, in principle, be a function of wave amplitude. Accordingly, one of the primary objectives of the present study is to derive a relationship between wave period and wave amplitude based on the velocity potential equation.

Once the wave constants are expressed as functions of wave amplitude, the analysis can be extended to incorporate wave energy as the input parameter. In this framework, wave amplitude is first determined from the given wave energy, and the resulting amplitude is then used to calculate the corresponding wave constants.

II. WEIGHTED TAYLOR SERIES AND WEIGHTING COEFFICIENT.

A truncated Taylor series refers to a Taylor expansion that is retained only up to the first-order terms. This formulation constitutes the fundamental basis for the derivation of the governing equations in hydrodynamics. In this study, a weighted Taylor series is employed, which modifies the truncated Taylor series by introducing coefficients to the first-order terms to compensate for the omission of higher-order terms. Following Hutahaean (2025a), these coefficients are referred to as weighting coefficients.

For a function $f = f(x,t)$, where x denotes the horizontal coordinate and t represents time, the weighted Taylor series can be expressed as

$$f(x+\delta x, t+\delta t) = f(x,t) + \gamma_{t,2}\delta t \frac{\partial f}{\partial t} + \gamma_{x,2}\delta x \frac{\partial f}{\partial x} \quad \ldots(1)$$

where $\gamma_{t,2}$ and $\gamma_{x,2}$ are weighting coefficients. Their baseline values are $\gamma_{t,2} = 2.0$ and $\gamma_{x,2} = 1.0$. In this study, the values $\gamma_{t,2} = 1.998933$ and $\gamma_{x,2} = 0.998933$ are adopted.

For a function $f = f(x,z,t)$, where x denotes the horizontal coordinate, z the vertical coordinate, and t time, the weighted Taylor series becomes

$$f(x+\delta x, z+\delta z, t+\delta t) = f(x,z,t) + \gamma_{t,3}\delta t \frac{\partial f}{\partial t} + \gamma_{x,3}\delta x \frac{\partial f}{\partial x} + \gamma_{z,3}\delta z \frac{\partial f}{\partial z} \quad \ldots(2)$$

$\gamma_{t,3}$, $\gamma_{x,3}$ and $\gamma_{z,3}$ are weighting coefficients. Their baseline values are $\gamma_{t,3} = 3.0$, $\gamma_{x,3} = 2.0$ and $\gamma_{z,3} = 2.0$. In the present study, the values $\gamma_{t,3} = 3.098667$ and $\gamma_{x,3} = \gamma_{z,3} = 2.098667$.

The procedure for calculating these weighting coefficients is described in Hutahaean (2025a).

III. VELOCITY POTENTIAL EQUATION.

The velocity potential equation, obtained as a solution of the Laplace's Equation using the separation of variables method, as presented by Robert G. Dean (1991) is

$$\phi(x,z,t) = G\,(\cos kx + \sin kx)\cosh k(h+z)\sin\sigma t \quad \ldots(3)$$

At the characteristic point, where $\cos kx = \sin kx$, this equation reduces to,

$$\phi(x,z,t) = 2G\cos kx \cosh k(h+z)\sin\sigma t \quad \ldots(4)$$

In equations (3) and (4), $\phi(x,z,t)$ denotes the velocity potential, G, k and σ are wave constants. G represents the rate of energy transmission, k is the wave number defined as $k = \frac{2\pi}{L}$, L is the wavelength, σ is angular frequency, $\sigma = \frac{2\pi}{T}$, T is wave period, h is water depth.

Along the axis-x, the complete velocity potential in Eq. (3) consists of two functions $\cos kx$ *and* $\sin kx$. These functions attain equal values when $\cos kx = \sin kx = \frac{1}{2}\sqrt{2}$ (see Figure 1). The calculation of wave constants, such as the wave number k is performed at this point, where the equality holds for both functions. This point is therefore referred to as the characteristic point.

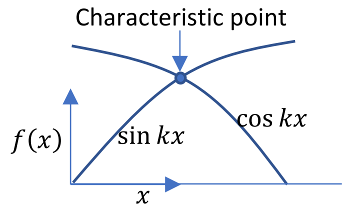

Fig (1). Characteristic point.

Along the time axis-t, Equation (3) contains the term $\sin\sigma t$. In subsequent derivations, expressions may involve both $\sin\sigma t$ and $\cos\sigma t$, or require the evaluation of one or both of these functions. For this reason, the calculations are carried out at the temporal characteristic point where $\sin\sigma t = \cos\sigma t = \frac{1}{2}\sqrt{2}$.

IV. RELATIONSHIP BETWEEN WAVE NUMBER AND WAVE AMPLITUDE

4.1. Review of the Wave Energy Equation

The total energy contained within one wavelength, according to Robert G. Dean (1991), is expressed as,

$$E = \frac{1}{8}\rho g H^2 L \qquad \ldots (5)$$

ρ is the water density, g is the gravitational acceleration, H is the wave height. For a sinusoidal wave $H = 2A$, A is wave amplitude, $L = \frac{2\pi}{k}$ is wavelength and k is wave number.

Equation (5) is written as,

$$E = \rho g \pi \left(\frac{A^2}{k}\right)$$

From the equation, for a given total energy E,

$$\frac{A^2}{k} = constant$$

This relation indicates an interdependence between the wave number k and A. A larger wave amplitude corresponds to a larger wave number k and therefore a shorter wavelength. Conversely, a smaller wave number (i.e., a longer wavelength) corresponds to a smaller wave amplitude.

Consequently, for a given input wave amplitude A, multiple wave configurations may arise. For example, waves W-I, W-II or W-III. Where η_{max} denotes the crest elevation; then $\eta_{max-I} > \eta_{max-II} > \eta_{max-III}$, and $L_I < L_{II} < L_{III}$, (See Figure 2).

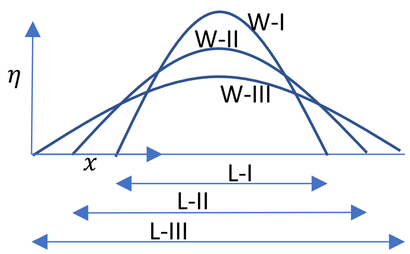

Fig (2). Possible wave profiles from wave input amplitude A.

4.2. Formulation of the Relationship Between k and A Using the Kinematic Free Surface Boundary Condition

Hutahaean (2023) formulated an equation relating the wave amplitude to the three wave constants. This equation is referred to as the wave amplitude function.

The formulation of the wave amplitude function employs the kinematic free surface boundary condition. By applying the weighted Taylor series formulation in Eq. (1), the Kinematic Free Surface Boundary Condition can be written as

$$w_\eta = \gamma_{t,2}\frac{\partial\eta}{\partial t} + \gamma_{x,2}u_\eta\frac{\partial\eta}{\partial x}$$

w is the vertical water particle velocity along the vertical axis-z, u s the horizontal water particle velocity along the horizontal x axis, w_η is the surface vertical water particle velocity and u_η is surface horizontal water particle velocity, $\eta = \eta(x,t)$ is water surface elevation.

Rearranging the equation yields the water surface elevation equation,

$$\gamma_{t,2}\frac{\partial\eta}{\partial t} = w_\eta - \gamma_{x,2}u_\eta\frac{\partial\eta}{\partial x} \qquad \ldots (6)$$

a. Wave Amplitude Function Type I

Using equation (3),

$$u(x,z,t) = -\frac{\partial\phi}{\partial x} = -Gk(-\sin kx + \cos kx)\cosh k(h+z)\sin\sigma t$$

$$u(x,\eta,t) = u_\eta(x,t) = -Gk(-\sin kx + \cos kx)\cosh k(h+\eta)\sin\sigma t \qquad \ldots (7)$$

$$w(x,z,t) = -\frac{\partial\phi}{\partial z} = -Gk(\cos kx + \sin kx)\sinh k(h+z)\sin\sigma t$$

$$w(x,\eta,t) = w_\eta(x,t) = -Gk(\cos kx + \sin kx)\sinh k(h+\eta)\sin\sigma t \qquad \ldots (8)$$

Substituting (7) and (8) to (6), to be integrated to time t,

$$\eta(x,t) = -\frac{2Gk}{\gamma_{t,2}}(\cos kx + \sin kx)\int \sinh k(h+\eta)\sin\sigma t\, dt + \frac{\gamma_{x,2}2Gk}{\gamma_{t,2}}(-\sin kx + \cos kx)\int \cosh k(h+\eta)\sin\sigma t\,\frac{\partial\eta}{\partial x}dt$$

$k(h+\eta) = kh\left(1+\frac{\eta}{h}\right)$, at deep water where $\frac{\eta}{h} \ll 1$, thus the fluctuation of $\frac{\eta}{h}$ towards time t is very small, that value $\frac{\eta}{h}$ is considered constant. Hence, $\sinh k(h+\eta)$ can be exempted from the integral. Meanwhile, the fluctuation

$\cosh kh\left(1+\frac{\eta}{h}\right)\frac{\partial\eta}{\partial x}$ towards time t becomes significantly smaller and it can be exempted from the integral. Under these approximations, the integration reduces to the time integration of $\sin\sigma t$ alone.

$$\eta(x,t)=\frac{Gk}{\gamma_{t,2}\sigma}(\cos kx+\sin kx)\sinh k(h+\eta)\cos\sigma t-\frac{\gamma_{x,2}Gk}{\gamma_{t,2}\sigma}(-\sin kx+\cos kx)\cosh k(h+\eta)\cos\sigma t\,\frac{\partial\eta}{\partial x}\quad ...(9)$$

η reaches the maximum value if $\frac{\partial\eta}{\partial x}=0$ and $\cos\sigma t=1$.

$$\eta_{max}=\frac{Gk}{\gamma_{t,2}\sigma}(\cos kx+\sin kx)\sinh k(h+\eta_{max})$$

$(\cos kx+\sin kx)$ reaches the maximum value if $\cos kx=\sin kx=\frac{1}{2}\sqrt{2}$.

$$\eta_{max}=\frac{\sqrt{2}Gk}{\gamma_{t,2}\sigma}\sinh k(h+\eta_{max})\quad ...(10)$$

At this stage, the value of η_{max} still depends on the term $\sinh k(h+\eta_{max})$ and, in particular, on the wave number k. Depending on the wavelength formed, for a given input wave amplitude A, a wave may be generated with

$\eta_{max}=\alpha_\eta A$, where $0<\alpha_\eta<1$.

$$\alpha_\eta A=\frac{\sqrt{2}Gk}{\gamma_{t,2}\sigma}\sinh k\left(h+\alpha_\eta A\right)$$

Or

$$A=\frac{\sqrt{2}Gk}{\alpha_\eta\gamma_{t,2}\sigma}\sinh k\left(h+\alpha_\eta A\right)\quad ...(11)$$

Equation (11) is referred to as the wave amplitude function Type I.

b. Wave Amplitude Function Type II.

In this section, the formulation is performed at the characteristic point, where $\cos kx=\sin kx$ .Using the velocity potential in equation (4).

$$u(x,z,t)=-\frac{\partial\phi}{\partial x}=2Gk\sin kx\cosh k(h+z)\sin\sigma t$$

$$u(x,\eta,t)=u_\eta(x,t)=2Gk\sin kx\cosh k(h+\eta)\sin\sigma t\quad ...(12)$$

$$w(x,z,t)=-\frac{\partial\phi}{\partial z}=-2Gk\cos kx\sinh k(h+z)\sin\sigma t$$

$$w(x,\eta,t)=w_\eta(x,t)\;-2Gk\cos kx\sinh k(h+\eta)\;\sin\sigma t\quad ...(13)$$

Substituting Equations (12) and (13) into Eq. (6), and integrating with respect to time t using the same procedure as in the previous section, while evaluating the expression at the characteristic point, yields

$$\eta(x,t)=\frac{2Gk}{\gamma_{t,2}\sigma}\cosh k(h+\eta)\left(\tanh k(h+\eta)+\gamma_{x,2}\frac{\partial\eta}{\partial x}\right)\cos kx\cos\sigma t$$

Since this expression represents a periodic wave, hence

$$\frac{2Gk}{\gamma_{t,2}\sigma}\cosh k(h+\eta)\left(\tanh k(h+\eta)+\gamma_{x,2}\frac{\partial\eta}{\partial x}\right)=constant$$

For a given input wave amplitude A, a periodic wave profile can be written as

$$\eta(x,t)=\alpha_\eta A\cos kx\cos\sigma t\quad ...(14)$$

$$\frac{\partial\eta}{\partial x}=-\alpha_\eta kA\sin kx\cos\sigma t$$

At the characteristic point,

$$\frac{\partial\eta}{\partial x}=-\frac{\alpha_\eta kA}{2}$$

Thus

$$A=\frac{2Gk}{\alpha_\eta\gamma_{t,2}\sigma}\cosh k\left(h+\alpha_\eta A\right)\left(\tanh k\left(h+\alpha_\eta A\right)-\frac{\gamma_{x,2}\alpha_\eta kA}{2}\right)\quad ...(15)$$

Equation (15) is referred to as the wave amplitude function Type II. A breaking-wave condition emerges from this equation when

$$\tanh k\left(h+\alpha_\eta A\right)-\frac{\gamma_{x,2}\alpha_\eta kA}{2}=0$$

Or,

$$\frac{H_b}{L_b}=\frac{2\tanh k\left(h+\alpha_\eta A\right)}{\gamma_{x,2}\alpha_\eta\pi}$$

or

$$\frac{H_b}{L_b}=\frac{2\tanh\theta\pi}{\gamma_{x,2}\alpha_\eta\pi}\quad ...(16)$$

The relationship between $k\left(h+\alpha_\eta A\right)$ and $\theta\pi$ is expressed in the following section.

c. Wave number equations

Wave amplitude at equation (11) should be the same as the wave amplitude at equation (15).

$$\frac{\sqrt{2}Gk}{\alpha_\eta\gamma_{t,2}\sigma}\sinh k\left(h+\alpha_\eta A\right)=\frac{2Gk}{\alpha_\eta\gamma_{t,2}\sigma}$$

$$\cosh k(h+\alpha_\eta A)\left(\tanh k(h+\alpha_\eta A)-\frac{\gamma_{x,2}\alpha_\eta kA}{2}\right)$$

$$k=\frac{(2-\sqrt{2})}{\gamma_{x,2}\alpha_\eta A}\tanh k(h+\alpha_\eta A) \quad \ldots(17)$$

At deep water, $\tanh k(h+\alpha_\eta A)\approx 1.0$. Where (Hutahaean (2023)),

$$k(h+\alpha_\eta A)=\theta\pi \quad \ldots(18)$$

This (17) becomes

$$k_0=\frac{(2-\sqrt{2})}{\gamma_{x,2}\alpha_\eta A_0}\tanh\theta\pi \quad \ldots(19)$$

The subscript 0 in k_0 and A_0 indicates that these quantities correspond to deep-water conditions. The parameter θ is referred to as the deep-water coefficient. For short waves, the approximation $\tanh\theta\pi\approx 1.0$ is commonly used.

As long as $\tanh\theta\pi\approx 1.0$, the calculated wave number is not highly sensitive to the value of θ. For example, the wave number obtained using $\theta=1.5$, for which $\tanh 1.5\pi=0.999839$ is very close to the value obtained using $\theta=3.0$, where $\tanh 3\pi=1.0$.

V. THE RELATIONSHIP BETWEEN WAVE CONSTANT G AND WAVE PERIOD AND WAVE AMPLITUDE.

The relationship between the wave constant G and the wave amplitude is formulated using the modified two-dimensional Euler momentum conservation equations proposed by Hutahaean (2025b), neglecting energy dissipation. The formulation consists of two governing equations.

The force balance in the horizontal $\boldsymbol{x}$ direction is

$$\gamma_{t,3}\frac{\partial u}{\partial t}-\frac{\gamma_{x,3}}{2}\frac{\partial uu}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \ldots(20)$$

The force balance in the vertical-z is

$$\gamma_{t,3}\frac{\partial w}{\partial t}-\frac{\gamma_{z,3}}{2}\frac{\partial ww}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g \quad \ldots(21)$$

u is horizontal water particle velocity, w is vertical water particle velocity, p is pressure to water particles, ρ is water mass density, and $\gamma_{t,3}$, $\gamma_{x,3}$ and $\gamma_{z,3}$ are weighting coefficients.

Equation (21) is first rearranged to obtain an expression for the pressure p. The resulting equation is then integrated along the vertical z axis while applying the dynamic free surface boundary condition, $p_\eta=0$. Subsequently, the resulting expression is differentiated with respect to the horizontal coordinate x,

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=\gamma_{t,3}\frac{\partial}{\partial x}\int_z^\eta\frac{\partial w}{\partial t}dz-\frac{\gamma_{z,3}}{2}\frac{\partial}{\partial x}(w_\eta w_\eta-ww)$$
$$+g\frac{\partial\eta}{\partial x}$$

The integration of the first term on the right-hand side is performed using the velocity potential formulation. From the velocity potential, the vertical velocity derivative with respect to time is,

$$\frac{\partial w}{\partial t}=-2Gk\sigma\cos kx\sinh k(h+z)\cos\sigma t$$

$$\int_z^\eta\frac{\partial w}{\partial t}dz=-2G\sigma\cos kx$$
$$(\cosh k(h+\eta)-\cosh k(h+z))\cos\sigma t$$

$$\frac{\partial}{\partial x}\int_z^\eta\frac{\partial w}{\partial t}dz=2Gk\sigma\sin kx$$
$$(\cosh k(h+\eta)-\cosh k(h+z))\cos\sigma t$$

Since,

$$u=2Gk\sin kx\cosh k(h+z)\sin\sigma t \quad \ldots(22)$$

$$\frac{\partial u}{\partial t}=2Gk\sigma\sin kx\cosh k(h+z)\cos\sigma t \quad \ldots(23)$$

Thus,

$$\frac{\partial}{\partial x}\int_z^\eta\frac{\partial w}{\partial t}dz=\left(\frac{\partial u_\eta}{\partial t}-\frac{\partial u}{\partial t}\right)$$

Thus

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=\gamma_{t,3}\left(\frac{\partial u_\eta}{\partial t}-\frac{\partial u}{\partial t}\right)-\frac{\gamma_{z,3}}{2}\frac{\partial}{\partial x}(w_\eta w_\eta-ww)+g\frac{\partial\eta}{\partial x}$$

Substituted to (20), yields

$$\gamma_{t,3}\frac{\partial u}{\partial t}-\frac{\gamma_{x,3}}{2}\frac{\partial uu}{\partial x}=-\gamma_{t,3}\left(\frac{\partial u_\eta}{\partial t}-\frac{\partial u}{\partial t}\right)$$
$$+\frac{\gamma_{z,3}}{2}\frac{\partial}{\partial x}(w_\eta w_\eta-ww)-g\frac{\partial\eta}{\partial x}$$

Identical terms cancel, yielding,

$$\gamma_{t,3}\frac{\partial u_\eta}{\partial t}-\frac{\gamma_{x,3}}{2}\frac{\partial uu}{\partial x}=\frac{\gamma_{z,3}}{2}\frac{\partial}{\partial x}(w_\eta w_\eta-ww)-g\frac{\partial\eta}{\partial x}$$

With $z=\eta$,

$$\gamma_{t,3}\frac{\partial u_\eta}{\partial t}-\frac{\gamma_{x,3}}{2}\frac{\partial u_\eta u_\eta}{\partial x}=-g\frac{\partial\eta}{\partial x} \quad \ldots(24)$$

Equations (22) and (23) are then evaluated at $z=\eta$. Using the free-surface elevation given by Eq. (14), and substituting these expressions into Eq. (24), the relation evaluated at the characteristic point becomes

$$\gamma_{t,3}2G\sigma\cosh\theta\pi-2\gamma_{x,3}G^2k^2\cosh^2\theta\pi=g\alpha_\eta A$$

Equation (11) can be rewritten as,

$$\sigma=\frac{\sqrt{2}Gk}{\alpha_\eta\gamma_{t,2}A}\sinh\theta\pi \quad \ldots(25)$$

Substituting it to equation (24)

$$\gamma_{t,3} 2G \frac{\sqrt{2}Gk}{\alpha_c \gamma_{t,2} A} \sinh\theta\pi \cosh\theta\pi$$
$$-2\gamma_{x,3} G^2 k^2 \cosh^2\theta\pi = g\alpha_\eta A$$

$$\left(\gamma_{t,3} \frac{\sqrt{2}k_0}{\alpha_c \gamma_{t,2} A_0} \sinh\theta\pi \cosh\theta\pi - \gamma_{x,3} k_0^2 \cosh^2\theta\pi\right) G_0^2$$
$$= \frac{g\alpha_\eta A_0}{2} \qquad \text{....(26)}$$

Substituting equation (19) to k,

$$\left(\frac{\gamma_{t,3}\sqrt{2}}{\gamma_{t,2}\gamma_{x,2}} - \frac{\gamma_{x,3}(2-\sqrt{2})}{\gamma_{x,2}^2}\right) G_0^2 = \frac{g\alpha_\eta^3 A_0^3}{2(2-\sqrt{2})\sinh^2\theta\pi}$$
$$\text{... (27)}$$

Index 0 shows the variable at deep water,This equation calculates G_0, and σ is calculated using equation (25).

VI. DETERMINING THE α_η VALUE

The analysis is performed using the water surface elevation equation (9) by computing the maximum elevation η_{max} and minimum elevation η_{min}. The appropriate value of α_η is obtained when $\eta_{max} - \eta_{min} = 2\,A_0$.

α_η value is obtained through the following steps.

a. Setting $\theta = 2.0$, or greater, where $\tanh\theta\pi \approx 1.0$, and $\cos\sigma t = 1$.

b. Setting $\alpha_\eta = 1.0$

c. Using the input wave amplitude A_0, k_0 is calculated using equation (19), G_0 is calculated using equation (27) and σ is calculated using equation (25).

d. Calculating the water depth $h = \frac{\theta\pi}{k_0} - \alpha_\eta A_0$

e. For $x = 0.625L - 2.0L$, η is calculated, and η_{max} and η_{min} are determined

$$\eta_0 = \alpha_\eta A_0(\cos k_0 x + \sin k_0 x)$$
$$\frac{\partial\eta}{\partial x} = \alpha_\eta k_0 A_0(-\sin k_0 x + \cos k_0 x)$$
$$\eta(x,t) = \frac{G_0 k_0}{\gamma_{t,2}\sigma}$$
$$(\cos k_0 x + \sin k_0 x)\sinh k_0(h + \eta_0)\cos\sigma t$$
$$-\frac{\gamma_{x,2} G_0 k_0}{\gamma_{t,2}\sigma}$$
$$(-\sin k_0 x + \cos k_0 x)\cosh k_0(h$$
$$+\eta_0)\cos\sigma t \frac{\partial\eta}{\partial x} \qquad \text{... (9)}$$

f. Checking, $\eta_{max} - \eta_{min}$,

- If $(\eta_{max} - \eta_{min}) < 2\,A_0$, α_η increases, repeat step c.
- If $(\eta_{max} - \eta_{min}) > 2\,A_0$, α_η decreases repeat step c.
- If $(\eta_{max} - \eta_{min}) \approx 2\,A_0$, the last α_η value is the α_η value being calculated.

This method yields $\alpha_\eta = 1.145011$. Figure (3) shows the results of the wave profile $\frac{\eta_{max}}{2A_0} = 0.7299$, where $A_0 = 1.20\,m$. In reference to Wilson's criteria (1963), this profile is cnoidal profile.

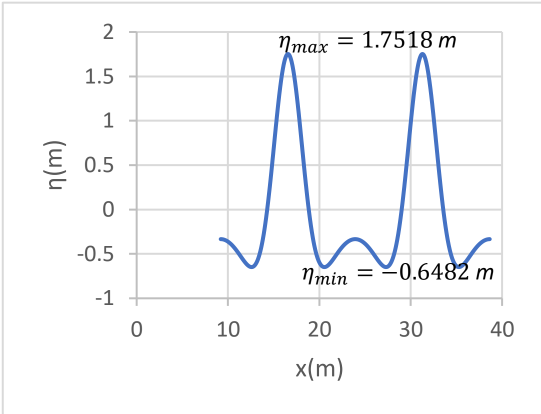

Fig (3) Resulting wave profile.

The corresponding wavelengths and wave periods for several wave amplitudes are presented in Table (1).

Table (1) Wavelength and Wave period for several wave amplitudes

$H = 2A$ (m)	L (m)	T (sec.)	$\frac{H}{L}$
1.2	7.367	4.247	0.163
1.6	9.823	4.904	0.163
2	12.278	5.482	0.163
2.4	14.734	6.006	0.163
2.8	17.19	6.487	0.163
3.2	19.646	6.935	0.163
3.6	22.101	7.355	0.163

Wave steepness obtained exceeds the critical wave steepness criteria proposed by Michell (1893), $\left(\frac{H}{L}\right)_{crit} = 0.142$, yet it is still smaller than the critical wave steepness criteria by Toffoli (2010), $\left(\frac{H}{L}\right)_{crit} = 0.170$.

Wiegel (1949, 1964) formulated the relationship between wave period and wave height as follows,

$$T_{Wieg} = 15.6\sqrt{\frac{H}{g}}\,\text{sec} \qquad \text{... (28)}$$

Where wave height $H = 2A$ (in meter).

Using the Pierson–Moskowitz spectrum, Silvester (1974) formulated a relationship between the wave period and the wave height as

$$T_{Silv} = 2.43\sqrt{\frac{H}{0.3048}}\,sec. \qquad \text{... (29)}$$

Wave height H in meter.

A comparison with the formulations of Robert L. Wiegel and Silvester is presented in Table (2), Fig. (4), and Table (3). As shown in Table (2), the largest wave period corresponds to T_{Wieg}, followed by T_{silv}, while the smallest values correspond to the wave period T obtained from the equation developed in the present study. Nevertheless, all three formulations exhibit a similar distribution pattern with respect to wave height, as illustrated in Fig. (3). The difference relative to T_{Wieg} reaches 22.167%, while the difference relative to T_{Silv} reaches 11.925% (Table 3). It should be noted, however, that both T_{Wieg} and T_{Silv} represent maximum possible values. Consequently, the wave period that typically occurs in practice is likely to be smaller than these maximum estimates.

Table (2). The comparison to Wiegel and Silvester's Equations.

H (m)	T (sec.)	T_{Wieg} (sec)	T_{Silv} (sec)
1.2	4.247	5.456	4.822
1.6	4.904	6.3	5.567
2	5.482	7.044	6.225
2.4	6.006	7.716	6.819
2.8	6.487	8.334	7.365
3.2	6.935	8.91	7.874
3.6	7.355	9.45	8.351

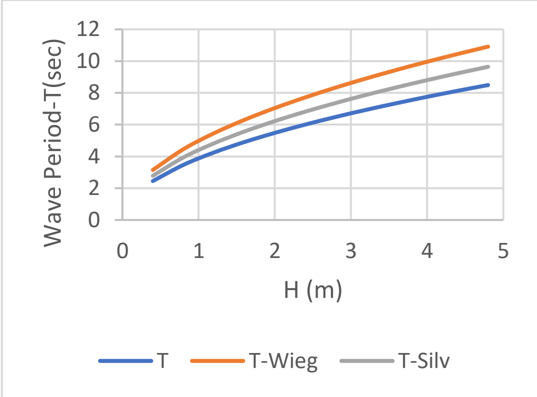

Fig (4) Wave period-T towards wave height H.

Table (3). Relative Differences from Wiegel and Silvester's Equations

H (m)	$\left\lvert\frac{T - T_{Wieg}}{T_{Wieg}}\right\rvert x100\%$	$\left\lvert\frac{T - T_{Silv}}{T_{Silv}}\right\rvert x100\%$
1.2	22.167	11.925
1.6	22.167	11.925
2	22.167	11.925
2.4	22.167	11.925
2.8	22.167	11.925
3.2	22.167	11.925
3.6	22.167	11.925

When adjustment is needed, for instance to T_{Wieg}, equation (19) is multiplied by $\alpha_k = 0.776163$, yielding equation (19) as,

$$k_0 = \frac{\alpha_k\left(2-\sqrt{2}\right)}{\gamma_{x,2}\alpha_\eta A_0}\tanh\theta\pi \qquad \text{... (30)}$$

The adjustment does not modify either α_η or wave profile. The results of the adjustments are presented in Table (4).

Table (4). Adjustment Using Wiegel's Equation.

H (m)	T (sec.)	T_{Wieg} (sec)	$\left\lvert\frac{T - T_{Wieg}}{T_{Wieg}}\right\rvert x100\%$
1.2	5.456	5.456	0
1.6	6.3	6.3	0
2	7.044	7.044	0
2.4	7.716	7.716	0
2.8	8.334	8.334	0
3.2	8.91	8.91	0
3.6	9.45	9.45	0

The adjustment increases the wavelength (Table 5).

Table (5). Wavelength at the adjusted wave period

$H = 2A$ (m)	L (m)	T (sec.)	$\frac{H}{L}$
1.2	9.492	5.456	0.126
1.6	12.656	6.3	0.126
2	15.819	7.044	0.126
2.4	18.983	7.716	0.126
2.8	22.147	8.334	0.126
3.2	25.311	8.91	0.126
3.6	28.475	9.45	0.126

With the revised wavelength, the resulting wave steepness becomes smaller than the critical wave steepness proposed by John Michell (1893).

Based on this result, the selection of the coefficient α_k can be determined by first specifying the most probable value of the wave steepness $\frac{H}{L}$. Once this representative steepness is established, the corresponding relationship between wave period and wave height can subsequently be derived.

VII. RELATIONSHIP BETWEEN WAVE AMPLITUDE AND WAVE ENERGY

6.1. Relationship between Wave Amplitude and Wave Energy.

The equation for wave energy over one wavelength (5) can be written as,

$$E = \rho g A^2 \frac{\pi}{k} \quad \text{....}(31)$$

Substituting k to (19),

$$k = \frac{(2-\sqrt{2})}{\gamma_{x,2}\alpha_\eta A}\tanh\theta\pi$$

$$E = \rho g \frac{\pi\gamma_{x,2}\alpha_\eta}{(2-\sqrt{2})\tanh\theta\pi}A^3$$

$$A_0 = \left(\frac{(2-\sqrt{2})\tanh\theta\pi\, E_0}{\rho g\pi\gamma_{x,2}\alpha_\eta}\right)^{1/3} \quad \text{...}(32)$$

Once the wave amplitude A_0 is obtained, the wave number k_0 can be calculated using Equation (19), the wave constant G_0 can be determined using Equation (27), and the angular wave frequency σ can be obtained from Equation (25). An example of the calculation results is presented in Table (6).

Table (6). The results of A_0, L_0 and T calculations and input energy.

E_0 (m)	$H_0 = 2A_0$ (m)	L_0 (m)	T (sec)
10000	1.099	6.749	4.064
20000	1.385	8.503	4.562
30000	1.585	9.733	4.881
40000	1.745	10.713	5.121
50000	1.88	11.54	5.315
60000	1.997	12.263	5.479
70000	2.103	12.91	5.621
80000	2.199	13.497	5.748
90000	2.287	14.038	5.862
100000	2.368	14.54	5.966

This method allows for the extension of the wave height, wave number, and wave period calculations to waves generated by ship motion, where the wave energy is assumed to originate from the ship's kinetic energy. Assuming a no-slip condition between the water and the ship hull, the wave energy can be approximated by

$$E = \rho\frac{V^2}{2g}D$$

V is the ship velocity and D is the ship draft.

VIII. APPLICATION TO LONG WAVES.

In this context, long waves refer to tsunamis and sneaker waves, which are sometimes also referred to as mini-tsunamis.

In deep water, the influence of wave amplitude on wavelength is very small, while the dominant parameter is the water depth. In this case, deep water refers to the water depth at which the calculation begins or the depth at which the wave is initially generated.

The relationship between deep-water wavelength and deep-water depth for long waves can be expressed using the wave number conservation equation (18):

$$k_0(h_0 + \alpha_\eta A_0) = \theta\pi \quad \text{...}(18)$$

k_0 is deep water wave number, h_0 is deep water depth, A_0 is deep water wave amplitude, θ is deep water coefficient at short wave, $\tanh\theta\pi \approx 1$ is used at the long wave. Whereas, $\tanh\theta\pi \approx \theta\pi$ at 5 % margin of error applies at $\theta = 0.124$. From (17), yielding

$$k_0 = \frac{\theta\pi}{h_0 + \alpha_\eta A_0} \quad \text{...}(33)$$

Since $A_0 \ll h_0$, the final equation can be approached using the following equation

$$k_0 = \frac{\theta\pi}{h_0} \quad \ldots (34)$$

The deep-water wave number can therefore be calculated using either Equation (33) or Equation (34), while the constant G is determined using Equation (26). It should not be calculated using Equation (27), since the influence of θ becomes significant in the case of long waves. Subsequently, the wave period is calculated using Equation (25).

An example of calculation results using a deep-water wave amplitude of $A_0 = 0.50\ m$ for several values of deep-water depth h_0 is presented in Table (7).

For long waves, the coefficient $\alpha_\eta = 1.000399$ is used, resulting in a sinusoidal wave profile. Figure (5) illustrates the long-wave profile for $h_0 = 100.0\ m,\ A_0 = 0.50\ m$.

Table (7). Deep water wavelength to the corresponding wave period

h_0 (m)	L_0 (m)	T (sec.)
50	109.303	810.484
100	154.771	1616.94
150	189.634	2423.39
200	219.016	3229.84
250	244.898	4036.29
300	268.294	4842.74

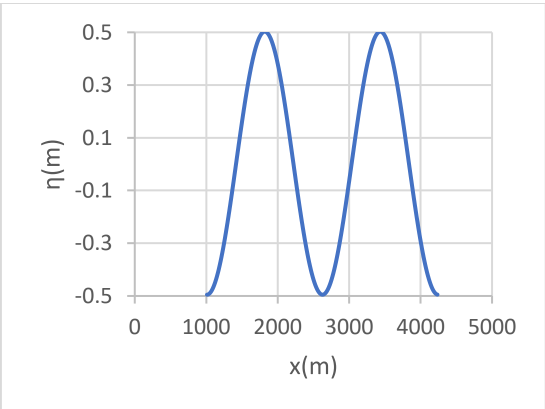

Fig (5). Longwave profile $A_0 = 0.50\ m, h_0 = 100.0\ m$

7.1. Calculation Using Input Energy.

Equation (28) can be rewritten as an expression for the wave amplitude:

$$A_0^2 = \frac{k_0}{\rho g \pi} E \quad \ldots (35)$$

where k_0 is calculated using Equation (34).

If a more accurate calculation is required, once A_0 has been obtained, the value of k_0 can be recalculated using Equation (33). Using this updated value of k_0, A_0 can then be recalculated accordingly. Subsequently, the wave period is determined using Equations (26) and (25).

Table (8) presents the calculation results for $E = 500000\ m$ at several values of deep-water depth h_0.

Table (8) Wave period and deep-water wave amplitude and wave length

h_0 (m)	A_0 (m)	T (sec)	L_0 (m)
100	0.502	154.867	1614.93
150	0.41	189.727	2421.01
200	0.355	219.104	3227.24
250	0.318	244.981	4033.54
300	0.29	268.373	4839.88
350	0.269	289.884	5646.25
400	0.251	309.904	6452.63
450	0.237	328.707	7259.02
500	0.225	346.491	8065.42

Table (9) presents the calculated deep-water wave amplitude, wavelength, and wave period at a deep-water depth of $h_0 = 100\ m$ for several values of input energy.

It can be observed that the wave period and wavelength remain nearly constant. This occurs because the resulting wave amplitude is very small compared with the water depth.

Table (9). Wave amplitude, wave period, and wavelength at $h_0 = 100\ m$.

E (m)	A_0 (m)	T (sec)	L_0 (m)
10000	0.071	154.95	1613.19
20000	0.101	154.945	1613.31
30000	0.123	154.94	1613.40
40000	0.142	154.937	1613.48
50000	0.159	154.933	1613.54
60000	0.174	154.93	1613.61
70000	0.188	154.928	1613.66
80000	0.201	154.925	1613.71
90000	0.213	154.923	1613.76
100000	0.225	154.921	1613.81

IX. CONCLUSION

The wave constant equations in this study were formulated analytically using conservation equations. Therefore, the resulting wave constants satisfy the fundamental conservation laws of hydrodynamics.

Some differences exist compared with empirical formulations, particularly in the relationship between wave amplitude (or wave height) and wave period. However, it should also be noted that the results obtained from empirical equations generally represent the maximum possible values that may occur.

In general, the wave constants obtained in this study provide reasonably good results and are sufficiently reliable for use in both wave dynamics analysis and engineering design of hydraulic or coastal structures.

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